Homework 5 Due Friday, February 23, 2024 at 11:59pm on gradescope

Academic honesty expectations: Same as on previous homeworks. We remind you that internet searches are not permitted.

1. (a) Find 2×2 matrices A and B such that $AB = O$ but $BA \neq O$. Similarly, find linear transformations $U, T : F^2 \to F^2$ such that $UT = T_0$ (the zero transformation) but $TU \neq T_0$.

   (b) Let V be a vector space, and let $T : V \to V$ be linear. Prove that $T^2 = T_0$ if and only if $R(T) \subseteq N(T)$.

2. Let V be a finite dimensional vector space and $T, U : V \to V$ be non-zero linear maps that satisfy $R(T) \cap R(U) = \{0_V\}$. Prove that T and U are linearly independent in $\mathcal{L}(V)$, the space of linear maps from V to V.

3. Let V, W, and Z be vector spaces, and let $T : V \to W$ and $U : W \to Z$ be linear.

   (a) Prove that if UT is one-to-one, then T is one-to-one. Must U also be one-to-one?

   (b) Prove that if UT is onto, the U is onto. Must T also be onto?

   (c) Prove that if U and T are bijections , then UT is also. (A bijection is a transformation that is both one-to-one and onto.)

4. Let $T : M_2(\mathbb{R}) \to P_3(\mathbb{R})$ be given by

$$T\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = (a+b) + dx + ax^2 + (c-b)x^3.$$

   Find an ordered basis β for $M_2(\mathbb{R})$ and an ordered basis γ for $P_3(\mathbb{R})$ so that $[T]_\beta^\gamma$ is the identity matrix I_4. Be sure to prove that the β and γ you choose are indeed bases.

5. (a) Let $V = \mathbb{R}^3$. Let $T_1 : \mathbb{R}^3 \to \mathbb{R}^3$ be given by $T_1(a, b, c) = (0, b, c)$. Let $T_2 : \mathbb{R}^3 \to \mathbb{R}^3$ be given by $T_2(a, b, c) = (0, a, b)$. Determine $[T_1]_\beta$ and $[T_2]_\beta$, where β is the standard ordered basis for $\mathbb{R}^3$. Determine bases for $N(T_i)$ and $R(T_i)$ for $i = 1, 2$..

   (b) Let V be a finite-dimensional vector space and $T \in \mathcal{L}(V)$. If rank(T)=rank(T^2), prove that $R(T) \cap N(T) = \mathbf{0}$. Deduce that $V = R(T) \oplus N(T)$. (Part (a) of this question is designed to help you think about part (b).)