MATH 235: HOMEWORK 4

DUE: FRIDAY, FEBRUARY 16 AT 11:59 PM ON GRADESCOPE
UNIVERSITY OF ROCHESTER, SPRING 2024

Follow the instructions on the course homework page to complete this assignment. Please adhere to the honesty policy detailed on the website. **Justify your answers fully.** Please list any collaborators.

Problem 1.
(1) Suppose that $p_0, p_1, p_2, \ldots, p_n$ are polynomials in $P_n(\mathbb{R})$ satisfying $p_i(2) = 0$. Show that $S = \{p_0, p_1, p_2, \ldots, p_n\}$ is not linearly independent.
(2) Let $V = \mathbb{C}^2$. Show that the vectors $v_1 = (1, i)$ and $v_2 = (i, 1)$ are a basis for V over $\mathbb{C}$. Determine scalars λ_1, λ_2 such that $\lambda_1 v_1 + \lambda_2 v_2 = (1 - i, 2 + i)$.
(3) Let W be the subspace of $M_{n\times n}(\mathbb{R})$ satisfying

$$W = \{A \in M_{n\times n}(\mathbb{R}) \mid \text{trace}(A) = 0\}.$$

Determine the dimension of W.

Problem 2.
(1) Determine subspaces U and W of $\mathbb{R}^3$ such that $\dim(U) = m$, $\dim(W) = n$, $m \geq n$, and both $\dim(U \cap W) < n$ and $\dim(U + W) < m + n$.
(2) Suppose that U and W are subspaces of a finite-dimensional vector space V and $V = U + W$. Show that $\dim(U) + \dim(W) \geq \dim(V)$ with equality if and only if $U \cap W = \{0\}$.
(3) If $\dim(U) + \dim(W) = \dim(V)$, does it follow that $V = U + W$? Prove this is true or provide a counter example.

Problem 3. Suppose V is a vector space over F, and $\dim(V) = n$. Find subspaces $W_1, W_2, \ldots, W_n$ of V such that

$$V = W_1 \oplus W_2 \oplus \cdots \oplus W_n.$$

Problem 4. Prove properties 1,2,3, and 4 of linear transformations on p. 65 (in both the 5th and 6th editions. Where it says See exercise 7.)

Problem 5. Let $W = \mathbb{R}_{>0}$ with addition and scalar multiplication defined in Q4 on HW1. Define $T : \mathbb{R}^2 \to W$ by $T(a, b) = 2^a 3^b$.
(1) Show that T is linear using property 2 that you proved in problem 4.
(2) Determine bases for the nullspace and range of T.