Problem (1)

(a) Let V be a vector space and W_1 and W_2 are subspaces of V. Prove that $W_1 \cup W_2$ is a subspace of V if and only if $W_1 \subseteq W_2$ or $W_2 \subseteq W_1$.
(b) For each of the following vectors $v \in V$ and subsets $S \subseteq V$, determine if $v \in \text{Span}(S)$.
  (i) $V = \mathbb{R}^4$, $S = \{(3,4,-2,7),(6,3,4,3)\}$, and $v = (1,-1,1,0)$.
  (ii) $V = P_3(\mathbb{R})$, $S = \{x^3+2x, 2x^3+1, x^2-x\}$, and $v = 7x^3 - 11x^2 - 5_x + 4$.

Problem (2) Read p. 22 where the *sum* $W_1 + W_2$ of two subspaces W_1 and W_2 of a vector space V is defined and where the *direct sum* $W_1 \oplus W_2$ of two subspaces W_1 and W_2 of a vector space V is defined.

(a) Show that $W_1 + W_2$ is a subspace of V.
(b) In other texts, we see the direct sum of W_1 and W_2 defined differently. That is, $W_1 + W_2$ is a direct sum, and we write $W_1 \oplus W_2$, if every element x in $W_1 + W_2$ can be written uniquely as a sum $x = u_1 + u_2$ where $u_1 \in W_1$ and $u_2 \in W_2$. Show that the two definitions are equivalent. (That is, show that if $W_1 \cap W_2 = \{\mathbf{0}\}$, then every $x \in W_1 + W_2$ can be written uniquely as a sum $x = u_1 + u_2$ where $u_1 \in W_1$ and $u_2 \in W_2$. Then show that, if every $x \in W_1 + W_2$ can be written uniquely, then $W_1 \cap W_2 = \{\mathbf{0}\}$.)
(c) We can defined the sum and direct sum of more than two subspaces. In the case of the direct sum, $W_1 \oplus W_2 \oplus \cdots \oplus W_n$, it is **not** sufficient to require that $W_i \cap W_j = \{\mathbf{0}\}$ for each $1 \leq i, j \leq n$ with $i \neq j$ in order to get the uniqueness property described in (b) . We need the stronger condition that

$$W_j \cap \sum_{i \neq j} W_i = \{\mathbf{0}\}.$$

Show this is true. First provide an example of a vector space V and three subspaces W_1, W_2, W_3 such that $W_1 \cap W_2 = \{\mathbf{0}\}$, $W_1 \cap W_3 = \{\mathbf{0}\}$, and $W_2 \cap W_3 = \{\mathbf{0}\}$, but there is a vector x such that x can be expressed in more than one way as $x = u_1 + u_2 + u_3$, with $u_1 \in W_1$, $u_2 \in W_2$, and $u_3 \in W_3$. Then show that, if the stronger condition is true, uniqueness follows.

Problem (3) Let $\mathcal{F}(\mathbb{R}, \mathbb{R})$ denote the space of functions from $\mathbb{R}$ to $\mathbb{R}$.

(a) A function $f : \mathbb{R} \to \mathbb{R}$ is *even* (resp. *odd*) if $f(-x) = f(x)$ (resp. $f(-x) = -f(x)$). Let $\mathcal{F}_{even}(\mathbb{R}, \mathbb{R})$ and $\mathcal{F}_{odd}(\mathbb{R}, \mathbb{R})$ denote the collections of even and odd functions respectively, from $\mathbb{R}$ to $\mathbb{R}$. Show that $\mathcal{F}_{even}(\mathbb{R}, \mathbb{R})$ and $\mathcal{F}_{odd}(\mathbb{R}, \mathbb{R})$ are subspaces of $\mathcal{F}(\mathbb{R}, \mathbb{R})$
(b) Show that $\mathcal{F}(\mathbb{R}, \mathbb{R}) = \mathcal{F}_{even}(\mathbb{R}, \mathbb{R}) \oplus \mathcal{F}_{odd}(\mathbb{R}, \mathbb{R})$. (Hint: Suppose $f \in \mathcal{F}(\mathbb{R}, \mathbb{R})$, and consider $h(x) = f(x) + f(-x)$.)

Problem (4)

Let V be a vector space and W a subspace of V. Given $x \in V$, the set $x+W := \{x+y : y \in W\}$ is called the *coset of* W *containing* x. (For example, let $V = \mathbb{R}^3$. Let $W = \{(x, y, 0) \mid x, y \in \mathbb{R}\}$. Let $x = (0, 0, 1)$. Then $x + W$ would be $\{(x, y, 1) \mid x, y \in \mathbb{R}\}$; geometrically, this is the horizontal plane through the point x.)

(a) If x is not an element of W, can $-x$ be an element of W?
(b) If $x + W = W$, what can you conclude about x? What must be true of x for $x + W$ to be a subspace of V?
(c) Prove that for any $h \in W$, $(x + h) + W = x + W$.
(d) Show that $x + W = x' + W$ if and only if $x - x'$ is in W. (Hint: It may help to use parts b and c)
(e) Prove that two cosets are either disjoint or else they are equal.