Homework 13

Due Sunday, April 28, 2024 at 11:59pm on gradescope

Academic honesty expectations: Same as on previous homeworks. We remind you that internet searches are not permitted.

1. Let V be a finite dimensional vector space, and let $T, U : V \longrightarrow V$ be linear operators.

   (a) Let $W_1, \dots, W_n$ subspaces of V that are invariant under both T and U (that is, $T(W_i) \subseteq W_i$ and $U(W_i) \subseteq W_i$ for each i). Suppose that

   $$W_1 + \cdots + W_n = V.$$

   Show that if $T_{W_i} U_{W_i} = U_{W_i} T_{W_i}$ for each i, then $TU = UT$.

   (b) Suppose that T is diagonalizable. Let $\lambda_1, \dots, \lambda_k$ be the eigenvalues of T and for each i, let $E_{\lambda_i} = \{v \in V \mid T(v) = \lambda_i v\}$. Show that if $U(E_{\lambda_i}) \subseteq E_{\lambda_i}$ for each i, then $TU = UT$. (Last week you showed the other direction, that if $TU = UT$ then $U(E_{\lambda_i}) \subseteq E_{\lambda_i}$.)

2. Let V be a vector space and let $T : V \longrightarrow V$ be a linear operator. Let $v \in V$ and let W be the T-cyclic space generated by v (that is, W is the span of $\{v, T(v), \dots, T^n(v), \dots\}$). Show that if Z is a T-invariant subspace of V that contains v, then Z also contains W.

3. Let $T \in \mathcal{L}(P_3(\mathbb{R}))$ be defined as $T(f(x)) = f''(x)$.

   (a) Determine a basis β_{W_1} for W_1, the cyclic subgroup generated by $f(x) = x^3$. Find $[T_{W_1}]_{\beta_{W_1}}$ and the characteristic polynomial of T_{W1}.

   (b) Choose a vector generating a cyclic subgroup W_2 such that $P_3 = W_1 \oplus W_2$. Repeat the same steps as in (a) for your W_2.

   (c) Now let β be the standard ordered basis for P_3. Determine $[T]_\beta$ and conclude that the characteristic polynomial of T is the product of the characteristic polynomials of T_{W_1} and T_{W_2}.

4. (a) Apply the Gram-Schmidt process to the set $\{(1,0,1),(0,1,1),(1,2,1)\}$ in $\mathbb{R}^3$ to obtain an orthogonal basis.

   (b) Apply the Gram-Schmidt process to the set $\left\{\begin{pmatrix} 2 & 2 \\ 2 & 1 \end{pmatrix}, \begin{pmatrix} 11 & 4 \\ 2 & 5 \end{pmatrix}, \begin{pmatrix} 4 & -12 \\ 3 & 16 \end{pmatrix}\right\}$ in $M_{2\times 2}(\mathbb{R})$ to obtain an orthogonal set with the same span.

5. Let V be an inner product space over $\mathbb{R}$.

   (a) Show that if x and y are orthogonal vectors in V, then $||x||^2 + ||y||^2 = ||x+y||^2$. Deduce the Pythagoerean Theorem in $\mathbb{R}^2$.

   (b) Show that x and y are orthogonal if and only if $||x|| \leq ||x + ay||$ for any $a \in \mathbb{R}$. [Hint: One direction of the proof is easy. For the other, choose a carefully as in the proof of the Cauchy-Schwarz inequalty.]

6. (a) Let W be a subset of a finite-dimensional inner product space V. Prove that W is a subspace if and only if $(W^\perp)^\perp = W$.

   (b) Let V be the vector space of all sequencesσ of real numbers such that $\sigma(n) = 0$ for all but finitely many n. (That is, V is the set of sequences that are eventually zero.) For $\sigma, \mu \in V$ define an

$$\langle \sigma, \mu \rangle = \sum_{n=1}^{\infty} \sigma(n)\mu(n).$$

      i. Prove that $\langle \cdot, \cdot \rangle$ defined above is an inner product.

      ii. Let $e_n = (0,0,\ldots,0,1,0,\ldots)$ be the vector which has 1 in the nth position and zero elsewhere. Prove that $\{e_1, e_2, \ldots\}$ is an orthonormal basis for V.

      iii. Let $\sigma_n = e_1 + e_n$ and let $W = \text{span}\{\sigma_n : n \geq 2\}$.

         A. Show that $e_1 \notin W$, so $W \neq V$.

         B. Prove that $W^\perp = \{0\}$ and conclude that $W \neq (W^\perp)^\perp$.