Homework 11 for S24 Math 235 Due Friday, April 12 at midnight on gradescope.

Follow the instructions on the course homework page to complete this assignment. Please adhere to the honesty policy detailed on the website. **Justify your answers fully.**

1. Use Cramer's Rule to evaluate the following system of linear equations:

$$\begin{aligned} 2x + y - 3z &= 1 \\ x - 2y + z &= 0 \\ 3x + 4y - 2z &= -7 \end{aligned}$$

2. This problem develops a way to invert a matrix using its *classical adjoint.* and Cramer's rule. Let c_{ij} denote the cofactor of the row i, column j entry of a matrix A. Let C be the transpose of the matrix whose ijth entry is c_{ij}. That is, $C_{ij} = c_{ji}$. Then C is the *classical adjoint* of A.

   (a) Show that if B is the matrix obtained from A by replacing column k with e_j, then $\det(B) = c_{jk}$.

   (b) Show that for $1 \leq j \leq n$,

$$A \begin{pmatrix} c_{j1} \\ c_{j2} \\ \vdots \\ c_{jn} \end{pmatrix} = \det(A) e_j$$

   by using Cramer's rule to solve $Ax = e_j$.

   (c) Show that $AC = [\det(A)]I$, where C is the adjoint defined above.

   (d) Conclude that if $\det(A) \neq 0$, then $A^{-1} = [\det(A)]^{-1}C$.

   (e) Invert the following matrices using this method.

$$A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \text{ and } B = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 2 & 0 \\ -1 & 0 & 4 \end{pmatrix}$$

3. Prove Theorem 5.3 on p. 250:

   Let $A \in M_{n \times n}(\mathbb{F})$.

   (a) The characteristic polynomial of A is a polynomial of degree n with leading coefficient $(-1)^n$.

   (b) A has at most n distinct eigenvalues.

4. (a) Suppose that $\lambda \in \mathbb{R}$ is an eigenvalue of $A \in M_{2\times 2}(\mathbb{R})$ of algebraic multiplicity 2. Show that if A is diagonalizable, then A must be diagonal.

   (b) Let $A \in M_{n\times n}(F)$ and $r \in F$. Suppose $\sum_{j=1}^{n} A_{ij} = r$ for each $i = 1, 2, \cdots, n$. Show that r is an eigenvalue of A. Give an eigenvector.

5. Let T be the linear operator on $M_{n\times n}(\mathbb{R})$ given by $T(A) = A^T$.

   (a) Show that ± 1 are the only eigenvalues of T. (Hint: Use the definition of eigenvalues and vectors rather than trying to determine the characteristic polynomial.)

   (b) For each eigenvalue, describe the corresponding eigenvectors.

   (c) Find an ordered basis for $M_{2\times 2}(\mathbb{R})$ such that $[T]_\beta$ is diagonal.

   (d) Generalize (c) to $n \times n$.