Homework 1 due January 26, 2024 at 11:59pm on gradescope.

Honesty policy for homework: The use of online resources such as chegg, online forums such as stack exchange, strange new chatbots or similar resources is strictly prohibited. You may discuss these problems with other students and/or your instructor, but you must write up your homework on your own. If the distinction between collaboration and copying is unclear to you, please consult your instructor.

(1) Fields

(a) Write the complex number $\dfrac{1}{a+bi}$ as $c + di$, where $c, d \in \mathbb{R}$.

(b) Let $\mathbb{Z}_n$ be the set $\{0, 1, 2, \ldots, n-1\}$ with the operations $\cdot$ and $+$ calculated modulo n. That is, for $x, y \in \mathbb{Z}_n$, $x + y$ in $\mathbb{Z}_n$ is equal to the remainder r of $x + y$ in $\mathbb{Z}$ upon division by n, and xy in $\mathbb{Z}_n$ is equal to the remainder r upon dividing xy in $\mathbb{Z}$ by n. Since $0 \leq r < n$, $\mathbb{Z}_n$ is closed under $\cdot$ and $+$ modulo n.

It turns out that, if $n = p$, a prime number, then $\mathbb{Z}_n$ is a field. We frequently write $\mathbb{F}_p$ rather than $\mathbb{Z}_p$. Create an addition table and a multiplication table for $\mathbb{F}_5$. Show that for each $a \in \mathbb{F}_5$, there is a b such that $a + b = 0$. Show that for each $a \neq 0$ in $\mathbb{F}_5$, there is a b in $\mathbb{F}_5$ such that $ab = 1$.

(2) Show that in a vector space V over a field F, if $a\mathbf{v} = \mathbf{0}$ where $a \in F$ and $\mathbf{v} \in V$, then $a = 0$ or $\mathbf{v} = \mathbf{0}$.

(3) Let S denote the set of ordered pairs of real numbers. Define a vector addition $\boxplus$ and scalar multiplication $\boxdot$ on S as follows.:

If (a_1, a_2) and $(b_1, b_2) \in S$ and $c \in \mathbb{R}$, then

$$(a_1, a_2) \boxplus (b_1, b_2) = (a_1 + b_1, a_2 b_2) \text{ and } c \boxdot (a_1, a_2) = (ca_1, a_2).$$

(a) Find an element in S that has the zero property defined in (VS 3).

(b) Find an element in S that has no additive inverse.

(c) From (b), we know that S a not vector space with these operations. Find an additional reason why S is not a vector space.

(4) Let $V = \mathbb{R}_{>0}$ be the set of strictly positive real numbers, and define a vector addition $\boxplus$ and scalar multiplication $\boxdot$ on V as follows:

- For u and v in V, let $u \boxplus v = uv$.
- For u in V and c in $\mathbb{R}$, let $c \boxdot u = u^c$.

Is V a vector space over $\mathbb{R}$ with these operations? Justify your answer by either verifying the vector space axioms, or providing a counterexample to one or more axioms.