Due Tuesday, April 12 at 11:59 PM. Please turn in on Gradescope. All chapter and exercise numbers refer to Friedberg, Insel, and Spence's *Linear Algebra*, 5th edition.

Please continue to follow policies on references, collaboration, and writing detailed in previous assignments.

(1) Suppose $A \in M_{n\times n}$ is a diagonalizable matrix with exactly one eigenvalue λ. Prove that $A = \lambda I_n$.

(2) Show that every matrix in $M_{2\times 2}(\mathbb{C})$ is similar to a matrix of the form

$$\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} a & 1 \\ 0 & a \end{pmatrix}$$

in $M_{2\times 2}(\mathbb{C})$. Show by counterexample that the statement is false if $\mathbb{C}$ is replaced by $\mathbb{R}$.

(3) Recall the definition of a nilpotent matrix from Problem Set 7.

Definition. A matrix $N \in M_{n\times n}(F)$ is *nilpotent* if there exists a positive integer k such that $N^k = 0$.

In this exercise, you will prove in two different ways that the only eigenvalue of a nilpotent matrix is zero.

(a) Suppose $\lambda \neq 0$ is an eigenvalue of a matrix $A \in M_{n\times n}(F)$. Show that $A^k \neq 0$ for all $k \in \mathbb{N}$.

(b) Suppose $N \in M_{n\times n}(F)$ is nilpotent. Find the characteristic polynomial of N and compute its roots. (Hint: apply the result of Exercise 4(d) in Problem Set 7.)

(4) Ex. 5.1.18(a)–(c)

(5) Suppose $A \in M_{n\times n}(\mathbb{C})$ has eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_k$ with multiplicities $m_1, m_2, \dots, m_k$, respectively.

(a) Prove $\det A = \lambda_1^{m_1} \lambda_2^{m_2} \cdots \lambda_k^{m_k}$.

(b) Prove $\operatorname{tr} A = m_1\lambda_1 + m_2\lambda_2 + \cdots + m_k\lambda_k$.

(Hints: For part (a), consider the constant term of the characteristic polynomial of A. For part (b), consider the t^{n-1} term of the characteristic polynomial of A.)

(6) Suppose $A \in M_{n\times n}(F)$ is invertible.

(a) Suppose λ is an eigenvalue of A. Show that $\lambda \neq 0$ and λ^{-1} is an eigenvalue of A^{-1}.

(b) Prove that A^{-1} is diagonalizable if and only if A is diagonalizable.

(7) Let $A \in M_{n\times n}(F)$.

(a) Show that A and A^t have the same eigenvalues and that these eigenvalues have the same multiplicities.

(b) Let E_λ and E'_λ denote the λ-eigenspace of A and A^t, respectively. Show that for any eigenvalue λ, we have $\dim E_\lambda = \dim E'_\lambda$. (Hint: use the rank-nullity theorem.)

(c) Show that A is diagonalizable if and only if A^t is diagonalizable, and if A is diagonalizable, then A and A^t have the same diagonalization.

(8) Fix $r \in F$ and $A \in M_{n\times n}(F)$.

(a) Suppose $\sum_{j=1}^n a_{ij} = r$ for all $1 \leq i \leq n$. Show that r is an eigenvalue of A and find a corresponding eigenvector.

(b) Suppose $\sum_{i=1}^n a_{ij} = r$ for all $1 \leq j \leq n$. Show that r is an eigenvalue of A.