Due Friday, March 25 at 11:59 PM. Please turn in on Gradescope. All chapter and exercise numbers refer to Friedberg, Insel, and Spence's *Linear Algebra*, 5th edition.

Homework policies: You may work together with other students, but solutions must be written on your own without reviewing others' written work. The use of online solution guides, paid resources (e.g. Chegg), and forums (e.g. Math Stack Exchange) is prohibited. If you are struggling with a problem, please review the textbook, consult your classmates, come to office hours, or reach out to the instructors or TAs.

Comments on writing: Your solutions should be written in paragraph form with correct grammar, usage, punctuation, etc. If you introduce new mathematical notation to refer to a certain number, object, equation, etc., be sure to define it clearly. Finally, please write large enough and leave enough blank space so that your paper is easily read. It is much easier for the graders to review and comment on solutions that are spread over three pages than crammed into one.

(1) Ex. 3.2.6(e)–(f)

(2) Ex. 3.2.14

(3) Ex. 3.2.19

(4) A square matrix N is said to be *nilpotent* if $N^k = 0$ for some $k \in \mathbb{N}$.

   (a) Suppose $N \in M_{n\times n}(F)$ is nilpotent. Show that N is not invertible.

   (b) Suppose $P \in M_{n\times n}(F)$ is similar to a nilpotent matrix N. Show that P is also nilpotent.

   (c) Show that every strictly upper triangular square matrix is nilpotent. (A square matrix $A = \{a_{ij}\}$ is strictly upper triangular if $a_{ij} = 0$ whenever $i \geq j$.)

   (d) Conversely, suppose $N \in M_{n\times n}(F)$ is nilpotent. Prove that N is similar to a strictly upper triangular matrix. (Hint: consider the inclusions

   $$\{0\} = R(L_N^k) \subset R(L_N^{k-1}) \subset \cdots \subset R(L_N^2) \subset R(L_N) \subset F^n$$

   and construct an ordered basis β of F^n such that $[L_N]_\beta$ is strictly upper triangular.)

(5) Determine the matrix $A \in M_{4\times 6}(\mathbb{R})$ with reduced row echelon form

$$\begin{pmatrix} 1 & -4 & 0 & 7 & 0 & -3 \\ 0 & 0 & 1 & -2 & 0 & 2 \\ 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

and first, third, and sixth columns given by

$$\begin{pmatrix} 3 \\ -2 \\ 4 \\ 1 \end{pmatrix}, \quad \begin{pmatrix} 4 \\ -3 \\ 5 \\ 1 \end{pmatrix}, \quad \text{and} \quad \begin{pmatrix} 2 \\ -2 \\ 1 \\ 3 \end{pmatrix},$$

respectively.

(6) Let S be the subset of $P_3(R)$ given by

$$\begin{aligned} S = \{&1 - 2x + 2x^2 + 4x^3, \\ &3 - 5x + 6x^2 + 11x^3, \\ &2 - x + 4x^2 + 5x^3, \\ &-1 - 2x - x^2 + x^3, \\ &5 - 10x + 12x^2 + 23x^3\}. \end{aligned}$$

Find a subset of S that is a basis of $P_3(\mathbb{R})$.