Due Tuesday, Feb. 15 at 11:59 PM. Please turn in on Gradescope. All chapter and exercise numbers refer to Friedberg, Insel, and Spence's *Linear Algebra*, 5th edition.

Homework policies: You may work together with other students, but solutions must be written on your own without reviewing others' written work. The use of online solution guides, paid resources (e.g. Chegg), and forums (e.g. Math Stack Exchange) is prohibited. If you are struggling with a problem, please review the textbook, consult your classmates, come to office hours, or reach out to the instructors or TAs.

Comments on writing: Your solutions should be written in paragraph form with correct grammar, usage, punctuation, etc. If you introduce new mathematical notation to refer to a certain number, object, equation, etc., be sure to define it clearly. Finally, please write large enough and leave enough blank space so that your paper is easily read. It is much easier for the graders to review and comment on solutions that are spread over three pages than crammed into one.

(1) Suppose that $T : \mathbb{R}^2 \to \mathbb{R}^2$ is linear and that $T((1,2)) = (3,4)$ and $T((1,3)) = (0,1)$. Find $T((1,0))$. Is T one-to-one? Is T onto? Justify your answer.

(2) You are given a map T between vector spaces over the same field. For each map complete each of the following.

  (i) Show that T is linear.
  (ii) Decide whether T is one-to-one or not (with justification).
  (iii) Decide whether T is onto or not (with justification). If it is not onto, provide a vector in the codomain of T that is not in the range of T.

  (a) $T : P_3(\mathbb{R}) \to M_2(\mathbb{R})$ defined by $T(p) = \begin{pmatrix} p(0) & p'(0) \\ p''(0) & p'''(0) \end{pmatrix}$.
  (b) $T : \mathbb{R}^2 \to \mathbb{R}^3$ defined by $T((a,b)) = (a, b, a+b)$.
  (c) Let V be the vector space of sequences described in Example 5 of §1.2. Define $T : V \to V$ by

$$T(a_1, a_2, a_3, \ldots) = (a_2, a_3, \ldots).$$

  Notice that V is not finite-dimensional.

(3) Let $V = M_2(\mathbb{R})$.

  (a) Give an example of a linear map $T : V \to V$ such that $N(T) = R(T)$.
  (b) Give examples of *distinct* linear maps $T : V \to V$ and $U : V \to V$ such that $N(T) = N(U)$ and $R(T) = R(U)$.

(4) Let V and W be finite-dimensional vector spaces and $T : V \to W$ be linear.

  (a) Prove that if $\dim(V) < \dim(W)$, then T cannot be onto.
  (b) Prove that if $\dim(V) > \dim(W)$, then T cannot be one-to-one.

(5) Let V and W be finite-dimensional vector spaces, V_1 be a subspace of V, and $T : V \to W$ be linear. Show that $T(V_1)$ is a subspace of W. Apply the Rank-Nullity theorem to $T|_{V_1}$ to conclude

$$\dim(V_1) = \dim(N(T) \cap V_1) + \dim(T(V_1)).$$

$T|_{V_1}$ is read "T restricted to V_1" and means that we change the domain of the transformation T from V to V_1.

(6) Let V and W be vector spaces and $T : V \to W$ linear.

  (a) Prove that T is one-to-one if and only if T carries linearly independent subsets of V onto linearly independent subsets of W.

(b) Suppose that T is one-to-one and that S is a subset of V. Prove that S is linearly independent if and only if $T(S)$ is linearly independent.

(c) Suppose $\beta = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is a basis for V and T is one-to-one and onto. Prove that $T(\beta) = \{T(\mathbf{v}_1), \ldots, T(\mathbf{v}_n)\}$ is a basis for W.