This assignment will not be collected. However, included material will appear on the final exam. All chapter and exercise numbers refer to Friedberg, Insel, and Spence's *Linear Algebra*, 5th edition.

(1) Let $V = \mathbb{R}^n$ with the standard inner product. View elements in V as column vectors. Suppose $A \in M_{n\times n}(\mathbb{R})$.

(a) Show that $\langle x, y\rangle = x^t y$ for all $x, y \in V$.

(b) Show that $\langle Ax, y\rangle = \langle x, A^t y\rangle$ for all $x, y \in V$.

(c) Suppose A is symmetric with distinct eigenvalues λ and μ and corresponding eigenvectors $v \in E_\lambda$ and $w \in E_\mu$. Show that v and w are orthogonal. (Hint: substitute v and w for x and y in the result of part (b)).

(2) For each of the following vectors spaces V defined over $\mathbb{R}$, show that the given function is not an inner product. (This requires finding an example that violates one part of the definition of an inner product.)

(a) $V = M_{3\times 3}(\mathbb{R})$ and $\langle A, B\rangle = \operatorname{tr}(AB)$

(b) $V = \mathcal{C}^0([0,1])$ and $\langle f, g\rangle = \sqrt{\int_0^1 f(x)^2 g(x)^2\, dx}$

(c) $V = \mathbb{R}^3$ and $\langle x, y\rangle = x^t A y$, where $A = \begin{pmatrix} 0 & 2 & 3 \\ 2 & 0 & 5 \\ 3 & 5 & 0 \end{pmatrix}$

Exercises 3 and 4 are about orthogonal matrices.

Definition. A square matrix $A \in M_{n\times n}(\mathbb{R})$ is said to be **orthogonal** if the columns of A are an orthonormal subset of $\mathbb{R}^n$ with respect to the standard inner product.

(3) Suppose $A \in M_{n\times n}(\mathbb{R})$. Show that $A^t A = I$ if and only if A is an orthogonal matrix.

(4) Suppose $A \in M_{n\times n}(\mathbb{R})$ is an orthogonal matrix.

(a) Show $\det A = \pm 1$.

(b) Show $\|Ax\| = \|x\|$ for all $x \in \mathbb{R}^n$, where $\|\cdot\|$ is the standard norm on $\mathbb{R}^n$. (Hint: compute $\|Ax\|^2$ and apply part (b) of Exercise 1.)

(c) Conclude that the only possible (real) eigenvalues of A are ± 1.

(5) For each subspace W of the inner product space V, construct an orthogonal basis of $W^\perp$. Indicate whether or not your basis is orthonormal.

(a) $V = \mathbb{C}^4$, $\langle x, y\rangle = x^t \overline{y}$, and $W = \operatorname{span}\left\{\begin{pmatrix} 1 \\ 2 \\ 0 \\ 1-i \end{pmatrix}, \begin{pmatrix} 0 \\ 2+2i \\ 0 \\ 3 \end{pmatrix}\right\}$

(b) $V = P_2(\mathbb{R})$, $\langle f, g\rangle = \int_0^2 f(x)g(x)\, dx$, and $W = \operatorname{span}\{1\}$

(c) $V = M_{2\times 2}(\mathbb{C})$, $\langle A, B\rangle = \operatorname{tr}(B^* A)$, and $W = \operatorname{span}\left\{\begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix}, \begin{pmatrix} 0 & i \\ 1 & 0 \end{pmatrix}\right\}$

(6) Ex. 6.2.12