Due Tuesday, April 19 at 11:59 PM. Please turn in on Gradescope. All chapter and exercise numbers refer to Friedberg, Insel, and Spence's *Linear Algebra*, 5th edition.

Please continue to follow policies on references, collaboration, and writing detailed in previous assignments.

(1) Consider the matrix

$$A = \begin{pmatrix} 1 & -2 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 3 \end{pmatrix} \in M_{3\times 3}(F).$$

Find A^n for $n \in \mathbb{N}$. Your answer should be a single 3×3 matrix depending on n. (Hint: consult Example 7, p. 271.)

(2) Suppose T is a linear operator on a vector space V over a field F. Prove the following.

(a) If λ is an eigenvalue of T and E_λ is its eigenspace, then E_λ is T-invariant.

(b) If W is a T-invariant subspace of V, then W is $g(T)$-invariant for any polynomial $g(t)$ with coefficients in F.

(3) Ex. 7.1.3(a)–(c), but only find a JCF of each map. You do *not* need to construct bases. Your work should mention eigenvalues of each map and dimensions of null spaces necessary to construct J. However, you do not need to show the computation of the eigenvalues or these dimensions.

(4) Suppose a linear map $T : V \to V$ has characteristic polynomial given by

$$f(t) = (t-2)^4(t+5)^2.$$

State every possible JCF of T up to reordering of Jordan blocks (no explanation needed).

(5) Let $T : V \to V$ be a linear map with split characteristic polynomial. Determine whether each of the following sets of dimensions is possible. If it is possible, give a JCF of T (no explanation needed). If it is not possible, explain why not.

(a)
- $\dim V = 7$
- $\dim N(T-8I) = 2$
- $\dim N((T-8I)^2) = 4$
- $\dim N(T+6I) = 1$
- $\dim N((T+6I)^3) = 3$

(b)
- $\dim V = 9$
- $\dim N(T+3I) = 3$
- $\dim N(T-4I) = 3$
- $\dim N((T-4I)^2) = 5$
- $\dim N((T-4I)^5) = 6$

(c)
- $\dim V = 10$
- $\dim N(T-5I) = 4$
- $\dim N((T+5I)^2) = 4$
- $\dim N((T+5I)^4) = 5$
- $\dim N((T+5I)^5) = 6$

(6) Suppose $A, B \in M_{n\times n}(F)$ are similar matrices.

(a) Show that $\operatorname{rank}((A-\lambda I)^r) = \operatorname{rank}((B-\lambda I)^r)$ for all $\lambda \in F$ and $r \in \mathbb{N}$.

(b) Suppose both A and B have split characteristic polynomials. Use part (a) to show A and B have the same JCF (except for possible reordering of Jordan blocks).

(c) Suppose $C \in M_{k\times k}(F)$ and $D \in M_{\ell\times\ell}(F)$. Show that the block diagonal matrices

$$M = \begin{pmatrix} C & \\ & D \end{pmatrix} \quad \text{and} \quad N = \begin{pmatrix} D & \\ & C \end{pmatrix}$$

are similar. (Hint: construct a matrix Q whose entries are all 0 and 1 such that $QM = NQ$.)

After an induction argument, the claims in the previous exercise establish the following theorem.

Theorem. *Two matrices A and B with split characteristic polynomials are similar if and only if they have the same Jordan canonical forms except for possible reordering of Jordan blocks.*

Apply this theorem to complete the following exercise.

(7) Ex. 7.2.6