Due Tuesday, January 25 at 11:59 PM. Please turn in on Gradescope. All chapter and exercise numbers refer to Friedberg, Insel, and Spence's *Linear Algebra*, 5th edition.

Homework policies: You may work together with other students, but solutions must be written on your own without reviewing others' written work. The use of online solution guides, paid resources (e.g. Chegg), and forums (e.g. Math Stack Exchange) is prohibited. If you are struggling with a problem, please review the textbook, consult your classmates, come to office hours, or reach out to the instructors or TAs.

Comments on writing: Your solutions should be written in paragraph form with correct grammar, usage, punctuation, etc. If you introduce new mathematical notation to refer to a certain number, object, equation, etc., be sure to define it clearly. Finally, please write large enough and leave enough blank space so that your paper is easily read. It is much easier for the graders to review and comment on solutions that are spread over three pages than crammed into one.

(1) We showed in class that $\mathbb{R}^n$ is a vector space over the field $\mathbb{R}$. For each of the following questions, please justify your answer (responses without explanation will not receive credit). You may assume that addition and scalar multiplication are defined in the standard way for n-tuples.

  (a) Is $\mathbb{R}^n$ a vector space over $\mathbb{C}$?

  (b) Is $\mathbb{C}^n$ a vector space over $\mathbb{R}$?

  (c) Is $\mathbb{R}^n$ a vector space over $\mathbb{Q}$?

  (d) Is $\mathbb{R}^n$ a vector space over $\mathbb{Z}$?

(2) Determine whether each of the following sets V together with operations $\boxplus$ and $\boxdot$ form a vector space over $\mathbb{R}$. An affirmative proof should demonstrate all vector space axioms hold. A negative proof should provide a counterexample to at least one of the axioms.

  (a) $V = \mathbb{Z}$; $\boxplus$ and $\boxdot$ are the usual addition and multiplication in $\mathbb{R}$.

  (b) $V = \{(a_1, a_2) \mid a_i \in \mathbb{R}\}$; for $\mathbf{u} = (u_1, u_2)$ and $\mathbf{v} = (v_1, v_2)$ and $\lambda \in \mathbb{R}$, define

$$\mathbf{u} \boxplus \mathbf{v} = (u_1 + v_2, u_2 + v_1) \quad \text{and} \quad \lambda \boxdot \mathbf{u} = (\lambda u_1, \lambda u_2).$$

  (c) $V = [-1, 1]$; for $u, v \in [-1, 1]$ and $\lambda \in \mathbb{R}$, define

$$u \boxplus v = \begin{cases} -1 & \text{if } u + v \le -1 \\ u + v & \text{if } -1 \le u + v \le 1 \\ 1 & \text{if } 1 \le u + v \end{cases} \qquad \text{and} \qquad \lambda \boxdot u = \begin{cases} -1 & \text{if } \lambda u \le -1 \\ \lambda u & \text{if } -1 \le \lambda u \le 1 \\ 1 & \text{if } 1 \le \lambda u. \end{cases}$$

  (d) $V = \{f \in \mathcal{F}(\mathbb{R}, \mathbb{R}) \mid f(x) > 0\}$; for $f, g \in V$ and $\lambda \in \mathbb{R}$, define

$$(f \boxplus g)(x) = f(x)g(x) \quad \text{and} \quad (\lambda \boxdot f)(x) = |\lambda| f(x).$$

(3) For each of the following subsets S of vector spaces V, determine whether S is a subspace of V.

  (a) $V = \mathcal{F}(\mathbb{R}, \mathbb{R})$, the set of real-valued functions on $\mathbb{R}$, and $S = \{f \in V \mid \lim_{t \to \infty} f(t) = 0\}$.

  (b) V is the set of infinite sequences of real numbers and $S = \{\{a_n\} \in V \mid a_1 + 3a_2 \le 1\}$.

  (c) $V = \mathcal{C}^1(\mathbb{R})$, the set of continuously differentiable functions, and $S = \{f \in V \mid f'(t) + (f(t))^2 = 0\}$.

  (d) $V = \mathcal{F}(\mathbb{R}, \mathbb{R})$ and $S = \{f \in V \mid f(x + y) = f(x) + f(y) \text{ for all } x, y \in \mathbb{R}\}$.

(4) Ex. 1.3.17

(5) Ex. 1.3.23