MATH 235: HOMEWORK 3

DUE: FRIDAY, FEBRUARY 9 AT 11:59 PM ON GRADESCOPE
UNIVERSITY OF ROCHESTER, SPRING 2024

Follow the instructions on the course homework page to complete this assignment. Please adhere to the honesty policy detailed on the website.

Problem 1. Suppose S_1 and S_2 are subsets of a vector space V.

(1) Show that if $S_1 \subseteq S_2$, then $\text{span}(S_1) \subseteq \text{span}(S_2)$.
(2) Show that $\text{span}(S_1 \cap S_2) \subseteq \text{span}(S_1) \cap \text{span}(S_2)$.
(3) Show that $\text{span}(S_1 \cup S_2) = \text{span}(S_1) + \text{span}(S_2)$. (Use the definition for the sum of subspaces from Homework 1.)

Solution:

(1). Let $x \in$span(S_1). Then $x = a_1x_1 + \cdots + a_kx_k$ for some $a_i \in \mathbb{F}$ and a finite subset $\{x_1, x_2, \ldots, x_k\}$ of S_1. Since $S_1 \subseteq S_2$, $\{x_1, x_2, \ldots, x_k\} \subseteq S_2$. Hence $x \in$span(S_2). As x was arbitrarily chosen, $\text{span}(S_1) \subseteq \text{span}(S_2)$.

(2) This is very similar to (1).

(3) To show set equality, it's often easiest to do two subset proofs. So first let $x \in \text{span}(S_1 \cup S_2)$. Set x equal to a linear combination of vectors in the union. Show that linear combination is in $\text{span}(S_1) + \text{span}(S_2)$. Then let $x \in \text{span}(S_1) + \text{span}(S_2)$ and reverse the process.

Problem 2. .

(1) Show that a subset S of a vector space V is linearly independent if and only if every subset of S is linearly independent.
(2) Suppose v, w are vectors in a vector space V. Show that $\{v, w\}$ is linearly independent if and only if neither vector is a scalar multiple of the other.

Solution:

(1) We have a theorem stating that subsets of independent sets are independent. For the other direction, since $S \subseteq S$, the conclusion follows.

(2) (Using the inverse and contrapositive instead of the statement and its converse.) Suppose $\{v, w\}$ is linearly dependent. Then there are scalars a, b, not both zero such that $av + bw = \mathbf{0}$. Assume without loss of generality that $a \neq 0$. Then $v = \frac{-b}{a}w$. Suppose $v = aw$. Then $v - aw = \mathbf{0}$. Since $1 \neq 0$, this is a non-trivial representation of $\mathbf{0}$, so $\{v, w\}$ is dependent.

Problem 3. . Suppose $\{v_1, v_2, v_3, \ldots, v_m\}$ spans V. Does

$$\{v_1 - v_2, v_2 - v_3, v_3 - v_4, \ldots, v_{m-1} - v_m, v_m\}$$

also span V?

Solution: Naming the sets: $S_1 = \{v_1, v_2, v_3, \ldots, v_m\}$ and $S_2 = \{v_1 - v_2, v_2 - v_3, v_3 - v_4, \ldots, v_{m-1} - v_m, v_m\}$.

There are two approaches. One could show directly that any $x \in V$ is in the span of S_2. One would do this by setting x equal to a linear combinations of vectors in S_1 and then tweaking the coefficiants to write it as a linear combination of vectors in S_2. Another approach would be to make use of the theorem that if W is a subspace and a set S is a subset of W, then $\text{span}(S) \subseteq W$. For this approach, just show that each vector in S_1 is in the span of S_2. This would mean $V = \text{span}(S_1) \subseteq \text{span}(S_2)$. Specifically, v_1 is the sum of the vectors in S_2. Then find v_2, etc.

Problem 4. Suppose $S = \{v_1, v_2, v_3, \ldots, v_m\}$ is linearly independent in V.

(1) Suppose $\lambda \in F$ and $\lambda \neq 0$. Is $\{\lambda v_1, \lambda v_2, \lambda v_3, \ldots, \lambda v_m\}$ linearly independent?
(2) Suppose $w \in V$. Show that if $\{v_1 + w, v_2 + w, v_3 + w, \ldots, v_m + w\}$ is linearly dependent, then $w \in \text{span}(S)$.

Solution:
(1) Yes.
(2) Set p a non-trivial representation of zero and solve for w.

Problem 5. .

(1) Can $P_3(\mathbb{R})$ (the polynomials of degree at most 3 with coefficients in $\mathbb{R}$) have a spanning set containing no vectors of degree exactly 2? Why or why not?
(2) Let $p_k(x) = \sum_{i=0}^{k} x^i$. Show that $\{p_k(x) \mid 0 \leq k \leq n\}$ is a basis for $P_n(\mathbb{R})$.
(3) Let $S = \{1, \cos x, \cos^2 x, \sin^2 x, \cos(2x), x\} \subset \mathcal{F}(\mathbb{R}, \mathbb{R})$. Determine a subset of S that is a basis for $\text{span}(S)$.

Solution:
(1) Yes. Consider this basis for $\mathbb{R}^4 : \beta = \{(0,0,0,1), (0,0,1,1), (0,1,1,1), (1,1,1,1)\}$.
(2) In this case the basis you want to consider for $\mathbb{R}^{n+1}$ is

$$\beta = \{(1, 0, \ldots, 0), (1, 1, 0, \ldots, 0), \ldots, (1, 1, \ldots, 1).\}$$

To use P_n instead, set $\sum_{k=0}^{n} a_k p_k = 0$ and generate a system of equations. Showing that all coefficients must be zero gives independence. Since there are $n+1$ vectors, you don't have to prove spanning. Another approach would be to show that the standard basis for P_n is in the span of the p_k. It would be similar to problem 3.