MTH 235 Spring 2023

Midterm 2, March 28, 2023

NAME (please print legibly): ____

Your University ID Number: ____

Please circle your instructor's name: **Madhu** **Kleene**

- No notes or electronic devices are permitted during the exam.
- Full justification is required on all questions except the True/False. In particular, if you provide a counter-example, you must explain why your counter-example is appropriate.
- Please initial to indicate that you have read and understood these instructions.________

PLEASE COPY THE HONOR PLEDGE AND SIGN. (Cursive is not required).
I affirm that I will not give or receive any unauthorized help on this exam, and all work will be my own.

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YOUR SIGNATURE:____

1. (20 points) Let $V = P_2(\mathbb{R})$. Show that $\alpha = \{-2x, 1 + x + 2x^2, -6x + 3x^2\}$ and $\gamma = \{(1 - x, 2x, 3 + x + x^2\}$ are both bases for $P_2(\mathbb{R})$. Determine a matrix A that changes α to γ coordinates.

For α:

Set $A = \begin{bmatrix} 0 & 1 & 0 \\ -2 & 1 & -6 \\ 0 & 2 & 3 \end{bmatrix} \longrightarrow \begin{bmatrix} -2 & 1 & -6 \\ 0 & 1 & 0 \\ 0 & 2 & 3 \end{bmatrix} \to \begin{bmatrix} -2 & 1 & -6 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix} = A'$

The rank of A' is 3, so the columns are independent. Then the vectors of α are independent. Since $\dim(P_2(\mathbb{R})) = 3$, α is a basis.

For γ: Set $B = \begin{bmatrix} 1 & 0 & 3 \\ -1 & 2 & 1 \\ 0 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 3 \\ 0 & 2 & 4 \\ 0 & 0 & 1 \end{bmatrix} = B'$. The rank of B' is 3 as well, so γ is also a basis.

Solution (A): Reduce $(B|A)$ to $(I \mid B^{-1}A)$.

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 3 & 0 & 1 & 0 \\ -1 & 2 & 1 & -2 & 1 & -6 \\ 0 & 0 & 1 & 0 & 2 & 3 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 0 & 3 & 0 & 1 & 0 \\ 0 & 2 & 4 & -2 & 2 & -6 \\ 0 & 0 & 1 & 0 & 2 & 3 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 3 & 0 & 1 & 0 \\ 0 & 1 & 2 & -1 & 1 & -3 \\ 0 & 0 & 1 & 0 & 2 & 3 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & -5 & -9 \\ 0 & 1 & 0 & -1 & -3 & -9 \\ 0 & 0 & 1 & 0 & 2 & 3 \end{array}\right]$$

So $\left[I_{P_2(\mathbb{R})}\right]_\alpha^\gamma = \begin{bmatrix} 0 & -5 & -9 \\ -1 & -3 & -9 \\ 0 & 2 & 3 \end{bmatrix}$.

Solution (B): $\left[I_{P_2(\mathbb{R})}\right]_\alpha^\gamma = \left[I_{P_2(\mathbb{R})}\right]_\beta^\gamma \left[I_{P_2(\mathbb{R})}\right]_\alpha^\beta$ where β is the standard order basis.

That is, $[I]_\alpha^\gamma = B^{-1}A$. We determine B^{-1}.

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 3 & 1 & 0 & 0 \\ -1 & 2 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 0 & 3 & 1 & 0 & 0 \\ 0 & 2 & 4 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & -3 \\ 0 & 2 & 0 & 1 & 1 & -4 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & -3 \\ 0 & 1 & 0 & \frac{1}{2} & \frac{1}{2} & -2 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right]$$

B^{-1}

$$B^{-1}A = \begin{bmatrix} 1 & 0 & -3 \\ \frac{1}{2} & \frac{1}{2} & -2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ -2 & 1 & -6 \\ 0 & 2 & 3 \end{bmatrix} = \begin{bmatrix} 0 & -5 & -9 \\ -1 & -3 & -9 \\ 0 & 2 & 3 \end{bmatrix}$$

Solution (C):

$-2x = 0(1-x) - 2x + 0(3+x+x^2)$: column 1 $= \begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix}$

$1+x+2x^2 = -5(1-x) - 3(2x) + 2(3+x+x^2)$: column 2 $= \begin{pmatrix} -5 \\ 3 \\ 2 \end{pmatrix}$

$-6x+3x^2 = -9(1-x) - 9(2x) + 3(3+x+x^2)$: column 3 $= \begin{pmatrix} -9 \\ -9 \\ 3 \end{pmatrix}$

2. (20 points) Let $T : P_2(\mathbb{R}) \to M_{2\times 2}(\mathbb{R})$ denote the transformation given by

$$T(f) = \begin{pmatrix} f(1) & f(-1) \\ f(0) & -f(1) \end{pmatrix}.$$

Show that T is an isomorphism on to the subspace of trace zero 2×2 matrices and compute $[T]_\beta^\gamma$ where $\beta = \{1, x, x^2\}$ is the standard basis of $P_2(\mathbb{R})$ and $\gamma = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \right\}$

To show T is an isomorp

Let $f(x) = a + bx + cx^2$.
$f(1) = a+b+c$. If $T(f) = 0$, then $f(1) = -f(1)$, so $a+b+c=0$.
If $f(0) = 0$, then $a = 0$.
If $f(-1) = 0$, then $c - b = 0$. We have $b+c = b-c$, so $b=c=0$.
Then T has trivial kernel, so T is one-to-one. Since the dimension of the trace zero 2×2 matrices is 3, and $\dim(P_2) = 3$, T is both one-to-one and onto.

For the matrix:

$$T(1) = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \text{ so } [T(1)]_\gamma = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$T(x) = \begin{pmatrix} 1 & -1 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} - \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \text{ so } [T(x)]_\gamma = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}.$$

$$T(x^2) = \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \text{ so } [T(x^2)]_\gamma = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}.$$

Then $$[T]_\beta^\gamma = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 0 & 0 \end{bmatrix}.$$

3. (20 points)

Let $A = \begin{bmatrix} 1 & -2 \\ -1 & 3 \end{bmatrix}$.

(a) Find elementary matrices E_1, E_2 such that the product $E_1 E_2 A = I_2$. Determine A^{-1}.

(b) Find a transformation $T : \mathbb{R}^2 \to P_1(\mathbb{R})$ such that $[T]_\beta^\gamma = A$, where $\beta = \{e_1, e_2\}$ and $\gamma = \{1, x\}$. Find a transformation $U : P_1(\mathbb{R}) \to \mathbb{R}^2$ such that $U = T^{-1}$.

a.) $\begin{bmatrix} 1 & -2 \\ -1 & 3 \end{bmatrix} \xrightarrow{R_1+R_2 \to R_2} \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix} \xrightarrow{R_1+2R_2 \to R_1} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$

$E_2 = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$ $E_1 = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$

$\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & -2 \\ -1 & 3 \end{bmatrix} = I_2$, so $A^{-1} = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$

$= \begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix}$

b.) Since $\begin{bmatrix} 1 & -2 \\ -1 & 3 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

$T(e_1) = 1 - x$

(thm 2.6 says it's sufficient to find $T(\beta)$.)

$\begin{bmatrix} 1 & -2 \\ -1 & 3 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -2 \\ 3 \end{bmatrix}$

$T(e_2) = -2 + 3x$

Since $\begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$, $u(1) = (3,1)$

$\begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$, $u(x) = (2,1)$

4. (20 points)

Suppose a matrix A has reduced row echelon form B, where

$$B = \begin{bmatrix} 0 & 1 & -3 & 0 & 2 \\ 0 & 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

(a) Suppose columns 1, 2 and 4 of A are as follows:

$$c_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, c_2 = \begin{bmatrix} -2 \\ 5 \\ 1 \end{bmatrix}, c_4 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

Determine A. Determine the rank of A and the dimension of its nullspace. (This question continues on the next page.)

(a) To determine A:

In B, $c_3 = -3c_2$

$c_5 = 2c_2 - 2c_4$

then in A $c_3 = -3\begin{pmatrix} -2 \\ 5 \\ 1 \end{pmatrix} = \begin{bmatrix} 6 \\ -15 \\ -3 \end{bmatrix}$

$c_5 = 2\begin{pmatrix} -2 \\ 5 \\ 1 \end{pmatrix} - 2\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{bmatrix} -6 \\ 8 \\ 0 \end{bmatrix}$

$$A = \begin{bmatrix} 0 & -2 & 6 & 1 & -6 \\ 0 & 5 & -15 & 1 & 8 \\ 0 & 1 & -3 & 1 & 0 \end{bmatrix}$$

rank $(A) = 2$

nullity $(A) = 5 - 2 = 3$

(b) Find a basis for the nullspace of A.

(c) Find the solution set of the linear system $Ax = b$, where $b = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$.

(b)

$$B = \begin{bmatrix} 0 & 1 & -3 & 0 & 2 \\ 0 & 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$x_2 - 3s + 2t = 0$

$x_2 = 3s - 2t$

$x_4 - 2t = 0$

$x_4 = 2t$

Let $x_1 = r$, $x_3 = s$, $x_5 = t$, $r, s, t \in F$

$N(A) = \left\{ \begin{pmatrix} r \\ 3s-2t \\ s \\ 2t \\ t \end{pmatrix} \right\}$

If $v \in$ nullspace of B,

$v = r\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} + s\begin{pmatrix} 0 \\ 3 \\ 1 \\ 0 \\ 0 \end{pmatrix} + t\begin{pmatrix} 0 \\ -2 \\ 0 \\ 2 \\ 1 \end{pmatrix}$

This spanning set is a basis, as it has the appropriate number of vectors.

(c) By inspection, $Ae_4 = b$. So e_4 is a particular solution.

Then the solution set has the form $K = \left\{ \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \right\} + K_H$

(c) In A, column 4 satisfies $Ae_4 = b$. Then $\begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$ is a particular solution.

So $K = \left\{ \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \right\} + \text{span}\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 3 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -2 \\ 0 \\ 2 \\ 1 \end{pmatrix} \right\}.$

5. (20 points) True of False. You do not need to justify your answers. No partial credit will be awarded.

1. (2pts) If A, B are matrices such that AB is defined and $AB = I$, then $B = A^{-1}$.

   True ☐ False ☒ Only true if A, B are square.

2. (2pts) The solution set of a system of linear equations in n unknowns is a subspace of $\mathbb{R}^n$.

   True ☐ False ☒ True if homogeneous.

3. (2pts) A matrix and its reduced row echelon form have the same nullspace.

   True ☒ False ☐

4. (2pts) If $\dim V = \dim W$ there exists a unique isomorphism $\phi : V \to W$.

   True ☐ False ☒ The isomorphism is not unique.

5. (2pts) If A is a 2×5 matrix and B is a a 5×2 matrix, then AB can have rank 5.

   True ☐ False ☒ rank $(AB) \leq 2$.

6. (2pts) There exists a 2×2 system of linear equations over $\mathbb{R}$ such that the solution set is $S = \{(1,0),(0,1)\}$.

   True ☐ False ☒

   It is possible to have no solutions, one solution, or ∞-many.

7. (8 pts) True of False: For each of the following, decide if the statement is true or false. In each part, T is a linear operator on a finite dimensional vector space V, $\beta = \{\beta_1, \beta_2, \dots, \beta_n\}$ and $\alpha = \{\alpha_1, \alpha_2, \dots, \alpha_n\}$ are both ordered bases for V, and Q is the change of coordinates matrix that changes β coordinates into α coordinates.

   (a) $Q = [I_V]_\alpha^\beta$.

   True ☐ False ☒

   (b) The jth column of Q is $[\beta_j]_\alpha$.

   True ☒ False ☐

   (c) Q^{-1} necessarily exists and it is the change of coordinates matrix from β to α coordinates.

   True ☐ False ☒

   (d) $[T]_\beta$ and $[T]_\alpha$ are similar matrices.

   True ☒ False ☐