MTH 235

Midterm 1

February 27, 2024

Name: Solutions

Student ID: ______________________________

(If you do not know your SID, provide your netid.)

The Department of Mathematics adheres to the university's academic honesty policy. In addition, the following restrictions apply to this exam:

1. No phones, calculators or any other devices that could help you answer any part of this exam are permitted.
2. No notes or formula sheets or similar documents are permitted.

PLEASE COPY THE HONOR PLEDGE AND SIGN:

(Cursive is not required).

I affirm that I will not give or receive any unauthorized help on this exam, and all work will be my own.

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YOUR SIGNATURE:______________________________

1. (20 points) Answer each question T for true or F for false. You do not have to justify your answer.

IMPORTANT: The T/F will be graded as follows: You will get 2 pts for a correct response. You will get 0 points for no response. You will get -.25 pts for an incorrect response. (You will get a minimum of zero for this question–no negatives.)

1. The following subset S of $P_2(\mathbb{R})$ is linearly independent:

$$S = \{x^2 - 1, x + 11, -7x, 5 - x\}.$$

- ☐ True
- ☒ False
- ☐ No response

$\dim(P_2(\mathbb{R})) = 3.$

2. The following set S is a spanning set for $P_3(\mathbb{R})$:

$$S = \{x^3 - 5, 4x, 3 - 6x\}$$

- ☐ True
- ☒ False
- ☐ No response

$\dim(P_3(\mathbb{R})) = 4.$

3. The following set W is subspace of $\mathbb{R}^4$.

$$W = \text{Span}\left(\{(1,1,1,1), (1,-1,2,7), (5,4,3,1)\}\right).$$

- ☒ True
- ☐ False
- ☐ No response

The span of a set of vectors in V is always a subspace.

4. If W_1 and W_2 are subspaces of a vector space V, then $W_1 \cap W_2$ is also a subspace of V.

- ☒ True
- ☐ False
- ☐ No response

HW 1.

5. A subset W of a vector space V is a subspace of V if and only if $\mathrm{Span}(W) = W$.

   - ☒ True
   - ☐ False
   - ☐ No response

   If $\mathrm{span}(W) = W$, then W is the span of a set, so a subspace.

   If W is a subspace, it contains the span of all its subsets. So $\mathrm{span}(W) \subseteq W$. Since $W \subseteq \mathrm{span}(W)$, we have equality.

6. If S a linearly independent subset of a vector space V, then every subset of $\underline{\underline{\mathrm{Span}}}(S)$ is also linearly independent.

   - ☐ True
   - ☒ False
   - ☐ No response

7. The set of real numbers, $\mathbb{R}$, is a vector space over $\mathbb{R}$ of dimension 1.

   - ☒ True
   - ☐ False
   - ☐ No response

8. If W is the set of matrices $A \in M_{2\times 2}(\mathbb{R})$ such that $A^2 = A$, then W is a subspace of $M_{2\times 2}(\mathbb{R})$.

   - ☐ True
   - ☒ False
   - ☐ No response

   This set is not closed under addition or scalar multiplication.

9. Every subset of a linearly independent set must be linearly independent.

   - ☒ True
   - ☐ False
   - ☐ No response

10. Let S be a subset of a vector space V with $\dim(V) = n$. If S generates V, then S must contain at least n vectors.

    - ☒ True
    - ☐ False
    - ☐ No response

2. (20 points)

(a) Let $T : V \to V$ be a linear transformation on a vector space V. Let $W = \{v \in V \mid T(v) = 2v\}$. Show that W is a subspace of V.

$T(\vec{0}) = \vec{0} = 2\vec{0}$, so $\vec{0} \in W$.

Suppose $x, y \in W$. Then $T(x+y) = T(x) + T(y) = 2x + 2y = 2(x+y)$.

So W is closed under vector addition.

Suppose $x \in W$ and $\lambda \in \mathbb{F}$. Then $T(\lambda w) = \lambda T(w) = \lambda(2w) = 2(\lambda w)$.

So W is closed under scalar multiplication.

Hence W is a subspace.

(b) Let V be a vector space and $x, y \in V$. Show that the span of $\{x, x+y\}$ is equal to the span of $\{x, y\}$.

Since x, $x+y$ are both contained in span$\{x,y\}$, and span$\{x,y\}$ is a subspace, span$\{x, x+y\} \subseteq$ span$\{x,y\}$.
Since $y = x+y-x$, $\{x,y\} \subseteq$ span$\{x, x+y\}$. So span$\{x,y\} \subseteq$ span$\{x, x+y\}$.
So we have equality.

3. (20 points)

(a) Let V be a vector space, and let $T : V \longrightarrow V$ be a linear transformation. Show that if $N(T) = \{0\}$, then $N(T^2) = \{0\}$.

suppose that $N(T) = \{0\}$

and suppose that

$v \in N(T^2)$. Then $T^2(v) = 0$

so $T(T(v)) = 0$ so

$T(v) = 0$ (since $N(T) = \{0\}$)

so $v = 0$ (since $N(T) = \{0\}$)

(b) Let $T : P_2(\mathbb{R}) \longrightarrow \mathbb{R}^3$ be a linear transformation. Show that if $T(1+x) = 0$, then $R(T)$ cannot be all of $\mathbb{R}^3$.

If $T(x+1) = 0$, then

$N(T) \neq \{0\}$,

so $\dim N(T) \geq 1$

By Rank-Nullity,

$\dim R(T) + \dim N(T) = 3$.

since $\dim P_2(\mathbb{R}) = 3$

so $\dim R(T) \leq 2 < \dim \mathbb{R}^3$

so $R(T) \neq \mathbb{R}^3$

4. (10 points) Let $T : P_3(\mathbb{R}) \to M_{2\times 2}(\mathbb{R})$ be given by $T(f) = \begin{bmatrix} f''(1) & 0 \\ 0 & f(0) \end{bmatrix}$.

(a) Determine a basis for the nullspace of T.

(b) Determine a subset S of $P_3(\mathbb{R})$ such that $T(S)$ is a basis for the range of T.

(a) $N(T) = \{ f \in P_3(\mathbb{R}) \mid f''(1) = 0$ and $f(0) = 0 \}$.

Let $f(x) = a + bx + cx^2 + dx^3$.

$f'(x) = b + 2cx + 3dx^2$

$f''(x) = 2c + 6dx$

$f''(1) = 2c + 6d$.

Then, if $f \in N(T)$, $a = 0$ and $c = -3d$.

So $f(x)$ has the form $bx - 3dx^2 + dx^3$

$= \text{span}\{x, x^3 - 3x^2\}$.

As $x, x^3 - 3x^2$ are not scalar multiples of one another, $\{x, x^3 - 3x^2\}$ is a basis for $N(T)$.

(b). We can extend $\{x, x^3 - 3x^2\}$ to a basis for $P_3(\mathbb{R})$.

One such basis would be $\beta = \{x, x^3 - 3x^2, x^2, 1\}$. *

Then $S = \{1, x^2\}$ satisfies (b). To confirm,

$T(1) = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ and $T(x^2) = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}$.

* Since $T(\beta)$ always spans $R(T)$, and since $x, x^3 - 3x^2 \in N(T)$, then $T(S)$ must span $R(T)$. See the proof of the rank-nullity theorem.

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5. (10 points)

Let $V = \mathbb{R}_{>0}$ be the set of strictly positive real numbers, and define a vector addition $\boxplus$ and scalar multiplication $\boxdot$ on V as follows:

- For u and v in V, let $u \boxplus v = uv$.
- For u in V and c in $\mathbb{R}$, let $c \boxdot u = u^{2c}$.

Provide an example to show that V is not a vector space. (Be sure to use the appropriate symbols $\boxplus$ and $\boxdot$ where needed.)

① The unit property does not hold for V. Let $x \in V$.
$1 \boxdot x = x^2$. Specifically, let $x = 2$. then $1 \boxdot 2 = 4 \neq 2$.

② VS6 also fails.
$(ab) \boxdot x = x^{2ab}$
$a \boxdot (b \boxdot x) = a \boxdot x^{2b} = x^{4ab}$.
Specifically, $(2 \cdot 3) \boxdot 4 = 4^{12}$ and $2 \boxdot (3 \boxdot 4) = (4^6)^4 = 4^{24}$

Note: $\mathbb{R}_{>0}$ with the same addition and multiplication defined as $c \boxdot u = u^c$ is a vector space, and we have used it in the homework sets. Since only multiplication has changed for this set, VS1 thru VS4 will hold.

VS7 also holds:
$a \boxdot (x \boxplus y) = (xy)^{2a} = x^{2a} y^{2a} = a \boxdot x \boxplus a \boxdot y$

VS8 holds as well.
$(a+b) \boxdot x = x^{2(a+b)} = x^{2a} x^{2b} = a \boxdot x \boxplus b \boxdot x$.
this is an addition of scalars.

6. (20 points)

(a) Let U and W be two dimensional subspaces of $\mathbb{R}^3$. Show that if $U \neq W$, then $U + W = \mathbb{R}^3$.

Here is one way. If $U \neq W$ then there is a $u \in U$ such that $u \notin W$, since we cannot have $U \subseteq W$ if $U \neq W$ because $\dim U = \dim W$. Then $U+W$ contains W but is larger than W, so $\dim(U+W) > \dim W = 2$ so $\dim U+W = 3$, so $U+W = \mathbb{R}^3$

Another way: $\{u_1, u_2\}$ basis for U $\{w_1, w_2\}$ basis for W. Since $W \notin U$, some $w_i \notin \operatorname{span}(\{u_1, u_2\})$ but then $\{u_1, u_2, w_i\}$ is lin. indep so a basis for $\mathbb{R}^3$

(b) Show that $U \cap W \neq \{0\}$.

If $U \cap W = \{0\}$, then
$U + W = U \oplus W$ and
$\dim(U+W) = \dim(U) + \dim(W)$
as proved on HW. But then
$\dim(U+W) = 4 > 3$, a contradiction
so $U \cap W \neq \{0\}$.

Another way: $\{u_1, u_2\}$ basis for U
and $\{w_1, w_2\}$ basis for W.

$\{u_1, u_2, w_1, w_2\}$ is dependent

so $a_1 u_1 + a_2 u_2 + a_3 w_1 + a_4 w_2 = 0$
some $a_i \neq 0$

Then $a_1 u_1 + a_2 u_2 = -(a_3 w_1 + a_4 w_2)$
is $\neq 0$ and is in $U \cap W$