MTH 235

Midterm 2

April 2, 2024

Name: Solutions

Student ID: ____________________

(If you do not know your SID, provide your netid.)

The Department of Mathematics adheres to the university's academic honesty policy. In addition, the following restrictions apply to this exam:

1. No phones, calculators or any other devices that could help you answer any part of this exam are permitted.
2. No notes or formula sheets or similar documents are permitted.

PLEASE COPY THE HONOR PLEDGE AND SIGN:
(Cursive is not required).
I affirm that I will not give or receive any unauthorized help on this exam, and all work will be my own.

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YOUR SIGNATURE:______________________________

1. (20 points) Let $A = \begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix}$

(a) Determine elementary matrices $E_1$and E_2 such that $E_2 E_1 A = I_2$.

$$A = \begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix} \xrightarrow{R_2 - 2R_1} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \xrightarrow{R_1 - R_2} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$E_1 = \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix} \qquad E_2 = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$$

(b) Determine A^{-1}.

$$E_2 E_1 = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix} = \begin{bmatrix} 3 & -1 \\ -2 & 1 \end{bmatrix} = A^{-1}$$

(c) Find a basis γ for $\mathbb{R}^2$ such that A is equal to the change of basis matrix $[I_2]_\beta^\gamma$ where $\beta = \{e_1, e_2\}$. ($A = \begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix}$ as on the last page)

If $A = [I_2]_\beta^\gamma$, then $A^{-1} = [I_2]_\gamma^\beta$

So $[I_2]_\gamma^\beta = \begin{bmatrix} 3 & -1 \\ -2 & 1 \end{bmatrix}$.

This means the columns of A^{-1} are the β-coordinate vectors of the desired γ vectors.

Hence $\gamma = \{(3,-2), (-1,1)\}$.

2. (25 points) Let A be a matrix having reduced row echelon form

$$E = \begin{bmatrix} 1 & 2 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

(a) Determine whether or not $L_A : \mathbb{R}^5 \to \mathbb{R}^3$ is onto, or explain why you do not have enough information.

Since columns 1, 3, and 5 of E are $e_1, e_2,$ and e_3, they are independent. Hence E has rank 3. Since row operations preserve rank, we know A has rank 3. This makes $R(L_A)$ a subspace of $\mathbb{R}^3$ of dimension 3, so $R(L_A) = \mathbb{R}^3$. Then L_A is onto.

(b) Determine whether or not the second, fourth, and fifth columns of A are linearly independent or explain why do you not have enough information to do so.

We know that E has been obtained from A by a sequence of EROs. Suppose $M = E_k E_{k-1} \cdots E_2 E_1$ is the product of the elementary matrices we multiply A by to get E. That is $MA = E$. Suppose C_2, C_4, C_5 are the 2^{nd}, 4^{th} & 5^{th} columns of A. Then $MC_2 = \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix}$, $MC_4 = \begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix}$ and $MC_5 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$.

Suppose there are constants a_2, a_4, a_5 such that

$$a_2 C_2 + a_4 C_4 + a_5 C_5 = 0.$$

Then $M(a_2C_2 + a_4C_4 + a_5C_5) = 0$

That means $a_2 MC_2 + a_4 MC_4 + a_5 MC_5 = 0$.

Or $a_2 \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix} + a_4 \begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix} + a_5 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = 0$. Then $2a_2 = 0$, $-a_4 = 0$, and $a_5 = 0$.

Which means $a_2 = a_4 = a_5 = 0$. Then C_2, C_4, C_5 are independent.

(c) For your convenience, we will write down the information about A from the previous page: A is a matrix having reduced row echelon form

$$E = \begin{bmatrix} 1 & 2 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

Find a basis for the nullspace of A, or explain why you do not have enough information to do so.

The nullspace of E is equal to the nullspace of A, as $Ex=0$ and $Ax=0$ are equivalent systems.

Set $x_2 = s$ and $x_4 = t$, free parameters. Then

$x_1 + 2s = 0$ and $x_3 - t = 0$.

Finally, $x_5 = 0$.

So the nullspace of $E = \{(-2s, s, t, t, 0) \mid s, t \in \mathbb{F}\}$.

$= \operatorname{span}\{(-2, 1, 0, 0, 0), (0, 0, 1, 1, 0)\}$.

These vectors are clearly independent.

So a basis for $\operatorname{null}(A) = \{(-2, 1, 0, 0, 0), (0, 0, 1, 1, 0)\}$

(d) For your convenience, we will write down the information about A from the previous page: A is a matrix having reduced row echelon form

$$E = \begin{bmatrix} 1 & 2 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

Find a solution $x \in \mathbb{R}^5$ to the $Ax = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ or explain why you do not have enough information to do so.

We do not have enough information.

Suppose (as in part (b)), that $MA = E$.

Then, if $Ax = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$, $MAx = Ex = M\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$.

Since M is unknown to us, so is $M\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$,

So we're not able to solve $Ex = M\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ or, equivalently,

$Ax = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$.

3. (15 points) Let $\gamma = \{1+x, 2+x\}$ be an ordered basis for $P_1(\mathbb{R})$. Let T be a linear operator on $P_1(\mathbb{R})$. Suppose that

$$[T]_\gamma = \begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix}.$$

(a) Let $\beta = \{1, x\}$ be the standard ordered basis for $P_1(\mathbb{R})$. Find a matrix Q such that $Q[T]_\gamma Q^{-1} = [T]_\beta$

$$[T]_\beta = [I_{P_1(\mathbb{R})}]_\gamma^\beta [T]_\gamma [I_{P_1(\mathbb{R})}]_\beta^\gamma$$

Then $Q = [I_{P_1(\mathbb{R})}]_\gamma^\beta$ is the desired matrix.

$$Q = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}.$$

(b) Find $T(-3x+5)$.

$$[T]_\beta = Q[T]_\gamma Q^{-1}.$$

$$Q = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}, \quad \text{so } Q^{-1} = \frac{1}{1-2}\begin{bmatrix} 1 & -2 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 2 \\ 1 & -1 \end{bmatrix}$$

$$[T]_\beta = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} -1 & 2 \\ 1 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} -3 & 6 \\ 2 & -2 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 2 \\ -1 & 4 \end{bmatrix}$$

$$[T(-3x+5)]_\beta = \begin{bmatrix} 1 & 2 \\ -1 & 4 \end{bmatrix}\begin{bmatrix} 5 \\ -3 \end{bmatrix} = \begin{bmatrix} -1 \\ -17 \end{bmatrix}$$

$$T(-3x+5) = -1 - 17x.$$

4. (20 points) Suppose that $A \in M_{2\times 3}(\mathbb{R}) = \begin{pmatrix} 1 & 0 & 2 \\ 1 & 1 & 4 \end{pmatrix}$.

(a) Find a matrix B satisfying

$$AB = \begin{pmatrix} -2 & -1 \\ 1 & 14 \end{pmatrix}.$$

Row reducing A w/ the columns of $\begin{pmatrix} -2 & -1 \\ 1 & 14 \end{pmatrix}$

gives

$$\left(\begin{array}{ccc:cc} 1 & 0 & 2 & -2 & -1 \\ 1 & 1 & 4 & 1 & 14 \end{array}\right)$$

$$\left(\begin{array}{ccc:cc} 1 & 0 & 2 & -2 & -1 \\ 0 & 1 & 2 & 3 & 15 \end{array}\right)$$

so letting

$$B = \begin{pmatrix} -2 & -1 \\ 3 & 15 \\ 0 & 0 \end{pmatrix} \text{ works}$$

(b) Let U be set of all $C \in M_{3\times 2}(\mathbb{R})$ (three rows, two columns) such that

$$AC = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

(where A is $\begin{pmatrix} 1 & 0 & 2 \\ 1 & 1 & 4 \end{pmatrix}$, as before) Show that U is subspace of $M_{3\times 2}$ and calculate its dimension. Explain your answer carefully.

It is a subspace since

(i) $\begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} \in U$

(ii) $AC_1 = 0$ and $AC_2 = 0$ means $A(C_1 + C_2) = 0$

(iii) $AC = 0$ means $A\lambda C = \lambda AC = 0$

Now, we know A row reduces to

$\begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 2 \end{pmatrix}$ which has $\left\{ \begin{pmatrix} -2 \\ -2 \\ 1 \end{pmatrix} \right\}$

as a basis for its null space

so U has a basis of

$$\left\{ \begin{pmatrix} -2 & 0 \\ -2 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -2 \\ 0 & -2 \\ 0 & 1 \end{pmatrix} \right\}$$

so $\dim U = 2$

5. (20 points) IMPORTANT: The T/F will be graded as follows: You will get 2 pts for a correct response. You will get 0 points for no response. You will get -0.5 for an incorrect response. **This is different** from the scoring on midterm 1. (You will get a minimum of zero for this question–no negatives.)

1. $A^2 = I$ implies $A = I$ or $A = -I$.

   ☐ True
   ☒ False
   ☐ No response

   $$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

2. Let $T : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ be a linear transformation and let α and β be ordered bases for $\mathbb{R}^n$. Then we have $\left([T]_\alpha^\beta\right)^{-1} = [T^{-1}]_\alpha^\beta$

   ☐ True
   ☒ False
   ☐ No response

   $$\left([T]_\alpha^\beta\right)^{-1} = [T^{-1}]_\beta^\alpha$$

3. Every change of coordinate matrix is invertible.

   ☒ True
   ☐ False
   ☐ No response

   A change of coordinate matrix is the matrix of an identity transformation. For any transformation $[T]$ is invertible if and only if T is.

4. An elementary matrix is always square.

   ☒ True
   ☐ False
   ☐ No response

   An elementary matrix is obtained by a row operation on I_n.

5. The only entries in an elementary matrix are zeros and ones.

   ☐ True
   ☒ False
   ☐ No response

   Consider the matrix obtained by adding twice one row of I_n to another.

6. If B is a matrix that can be obtained by performing an elementary row operation on a matrix A, then A can be obtained by performing an elementary row operation on B.

   ☒ True
   ☐ False
   ☐ No response

   $EA = B \Rightarrow A = E^{-1}B$

   E^{-1} is also an elementary matrix.

7. If E is an elementary matrix, then $\det(E) = \pm 1$.

   ☐ True
   ☒ False
   ☐ No response

   Consider the matrix obtained by multiplying one row of I_n by 3.

8. Let $T : V \longrightarrow V$ be a linear transformation. Then if T is onto, it must also be one-to-one.

   ☐ True
   ☒ False
   ☐ No response

   This holds if V is finite-dimensional.

   A counterexample: $T: S \to S$ where S is the space of infinite sequences.

   $T(a_1, a_2, \dots) = (a_2, a_3, \dots)$ is onto, but not one-to-one.

   $T(a_1, a_2, \dots) = (0, a_1, a_2, \dots)$ is one-to-one, but not onto.

9. Regardless of of the specific operations you use to row reduce a matrix, you will arrive at the same reduced row echelon matrix.

   ☒ True
   ☐ False
   ☐ No response

   The RREF of A is unique.

10. The span of the columns of a matrix is the same as the span of the columns of its reduced row echelon form.

    ☐ True
    ☒ False
    ☐ No response

    Let $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$. Its column span is span$\left\{\begin{pmatrix} 1 \\ 1 \end{pmatrix}\right\}$.

    Its RREF form is $\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$. Its column span is span$\left\{\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right\}$.