MTH 235 Spring 2023

Final Exam, May 1, 2023

NAME (please print legibly): Solution Set ____________________

Your University ID Number: ____________________

Please circle your instructor's name: **Madhu** **Kleene**

- No notes or electronic devices are permitted during the exam.
- Full justification is required on all questions except the True/False. In particular, if you provide a counter-example, you must explain why your counter-example is appropriate.
- Please initial to indicate that you have read and understood these instructions.________

PLEASE COPY THE HONOR PLEDGE AND SIGN. (Cursive is not required).
I affirm that I will not give or receive any unauthorized help on this exam, and all work will be my own.

____

____

____

YOUR SIGNATURE:____________________

1. (16 points) Please choose true or false for the following questions. You do not need to justify your work, and partial credit will not be offered.

1. Let T be a linear operator on an n-dimensional vector space. Then there exists a polynomial $g(t)$ of degree n such that $g(T) = T_0$.

   ☑ TRUE ☐ FALSE

   Cayley-Hamilton Theorem.

2. A change-of-coordinates matrix is always invertible.

   ☑ TRUE ☐ FALSE

   $Q = [I_V]_\beta^\gamma$, and I_V is invertible.

3. There exists a linear operator T on $\mathbb{R}^n$ such that every non-zero $v \in \mathbb{R}^n$ is an eigenvector of T.

   ☑ TRUE ☐ FALSE

   The identity transformation: $\lambda = 1$.
   The zero transformation: $\lambda = 0$.

4. Every square matrix A satisfies $\det(AA^t) = \det(A^tA) = \det(A^2)$.

   ☒ TRUE ☐ FALSE

   $\det(A) = \det(A^T)$
   $\det(AB) = (\det A)(\det B)$

5. If U, W_1, W_2 are subspaces of a vector space V such that $W_1 + U = W_2 + U$, then $W_1 = W_2$.

   (Recall that $W_1 + W_2 = \{x_1 + x_2 | x_1 \in W_1, x_2 \in W_2\}$.)

☐ TRUE ☑ FALSE

Let $U = V = \mathbb{R}^3$, $W_1 = \text{span}\{(1,0,0)\}$, $W_2 = \text{span}\{(0,1,1)\}$.

6. If a matrix is diagonalizable, then it is invertible.

☐ TRUE ☑ FALSE

this is only true if all eigenvalues are non-zero.

7. A 2×2 matrix can have more than 3 eigenvectors.

☑ TRUE ☐ FALSE

Scalar multiples of eigenvectors are also eigenvectors.

8. Let $V = P_2(\mathbb{R})$. Then $\langle f, g \rangle = \int_0^1 f'(x)g(x)dx$ defines an inner product on V.

☐ TRUE ☑ FALSE

let $f(x) = 3$. then $\langle f, f \rangle = 0$, but $f \neq 0$.

2. (12 points)

Let $V = M_{2\times 2}(\mathbb{R})$ with the inner product $\langle A, B\rangle = \mathrm{tr}(B^T A)$.

Let $W = \mathrm{span}\left\{ \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & -1 \\ 2 & 1 \end{pmatrix} \right\}$.

(a) Determine an orthogonal basis for W. (This problem continues on the next page.)

$$v_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$v_2 = \begin{pmatrix} 1 & -1 \\ 2 & 1 \end{pmatrix} - \frac{\left\langle \begin{pmatrix} 1 & -1 \\ 2 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right\rangle}{\left\| \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right\|^2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$\mathrm{tr}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}^T \begin{pmatrix} 1 & -1 \\ 2 & 1 \end{pmatrix} = \mathrm{tr}\begin{pmatrix} 2 & 1 \\ 1 & -1 \end{pmatrix} = 1$$

$$\left\| \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right\|^2 = \left\langle \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right\rangle = \mathrm{tr}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}^2 = \mathrm{tr}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = 2$$

$$= \begin{pmatrix} 1 & -1 \\ 2 & 1 \end{pmatrix} - \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & \frac{-3}{2} \\ \frac{3}{2} & 1 \end{pmatrix}$$

$\beta = \left\{ \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & -3/2 \\ \frac{3}{2} & 1 \end{pmatrix} \right\}$ is an orthogonal basis.

(b) Find a vector in $W^\perp$. (Recall: $W^\perp = \{x \in V \mid \langle x, w\rangle = 0$ for every $w \in W\}$.)

$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \in W^\perp$. More generally, we

We need $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ such that $\operatorname{tr}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = 0$

so $b + c = 0$

Also $\operatorname{tr}\begin{pmatrix} 1 & \frac{3}{2} \\ -\frac{3}{2} & 1 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = 0$

So $a + \frac{3}{2}c + -\frac{3}{2}b + d = 0$

Substituting $b = -c$, $a + 3c + d = 0$, so $d = -a - 3c = -a + 3b$

Then a matrix in $W^\perp$ has the form $A = \begin{pmatrix} a & b \\ -b & -a+3b \end{pmatrix}$

So $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \in W^\perp$

Alternately, find a $w_3 \notin W$, such as $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, and use the Gram Schmidt process:

$$v_3 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} - \frac{\left\langle \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right\rangle}{\left\| \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right\|^2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} - \frac{\left\langle \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & -\frac{3}{2} \\ \frac{3}{2} & 1 \end{pmatrix} \right\rangle}{\left\| \begin{pmatrix} 1 & -\frac{3}{2} \\ \frac{3}{2} & 1 \end{pmatrix} \right\|^2} \begin{pmatrix} 1 & -\frac{3}{2} \\ \frac{3}{2} & 1 \end{pmatrix}$$

Calculations: ① $\left\langle \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right\rangle = 0$

② $\left\langle \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & -\frac{3}{2} \\ \frac{3}{2} & 1 \end{pmatrix} \right\rangle = 1$

③ $\left\| \begin{pmatrix} 1 & -\frac{3}{2} \\ \frac{3}{2} & 1 \end{pmatrix} \right\|^2 = 2 + 2\left(\frac{9}{4}\right) = \frac{13}{2}$

$$v_3 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} - \frac{2}{13}\begin{pmatrix} 1 & -\frac{3}{2} \\ \frac{3}{2} & 1 \end{pmatrix} = \begin{pmatrix} \frac{11}{13} & \frac{3}{13} \\ -\frac{3}{13} & -\frac{2}{13} \end{pmatrix}$$

3. (12 points) The following matrix A is not diagonalizable. Provide a Jordan cannonical matrix similar to A. You do not need to provide a corresponding basis, but you do need to justify your answer.

$$A = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

The characteristic polynomial is $p(t) = (1-t)^2(-t)^3$

$\underline{\underline{\lambda_1 = 1}}$ $\quad A - I = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 \end{pmatrix} \to \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \to \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$

rank $= 4$, so $\dim(E_{\lambda_1}) = 1$. $\dim(K_{\lambda_1}) = 2$, so a basis for K_{λ_1} contains only one eigenvector. The Jordan block for $\lambda_1 = 1$ is $\boxed{\begin{matrix} 1 & 1 \\ 0 & 1 \end{matrix}}$.

$\underline{\underline{\lambda_2 = 0}}$ $\quad A - 0I = A$, which has rank 3. So the $\dim(E_{\lambda_2}) = 2$.
Then a basis for K_{λ_2} has two eigenvectors. We get two Jordan blocks: $\boxed{0}\ \boxed{\begin{matrix} 0 & 1 \\ & 0 \end{matrix}}$

$$J = \left(\begin{array}{cc|c|cc} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right).$$

4. (12 points)

(a) Suppose that $T : \mathbb{R}^4 \to \mathbb{R}^2$ is a linear map satisfying

$$N(T) = \{(a_1, a_2, a_3, a_4) \in \mathbb{R}^4 : 3a_1 = a_4, a_2 = -a_3\}.$$

Prove or disprove: T is surjective.

$N(T) = \text{span}\{(1,0,0,3), (0,1,-1,0)\}$.

Then $\dim(N(T)) = 2$, so, by the dimension theorem, $\dim(R(T)) = 2$. Since $R(T) \subseteq \mathbb{R}^2$, and $\dim(\mathbb{R}^2) = 2$, $R(T) = \mathbb{R}^2$. Hence T is onto.

(b) Is there a linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^4$ such that

$$R(T) = \{(a_1, a_2, a_3, a_4) | a_1 = -a_2\}?$$

Give an example or explain why no such example exists.

If such a T exists, $R(T) = \text{span}\{(1,-1,0,0), (0,1,0,0), (0,0,1,0)\}$.

Then $\dim(R(T)) = 3 > \dim(\mathbb{R}^2)$. The dimension theorem says that $\dim(R(T))$ must be less than or equal to the dimension of the domain.

5. (12 points) Let $T \in \mathcal{L}(P_2(\mathbb{R}))$ be defined by $T(f) = f'(x)(x-1)$. If possible, determine a basis γ with respect to which $[T]_\gamma$ is diagonal. Also determine $[T]_\gamma$

Let $\beta = \{1, x, x^2\}$.

$[T]_\beta = \begin{bmatrix} 0 & -1 & 0 \\ 0 & 1 & -2 \\ 0 & 0 & 2 \end{bmatrix}$. Since $[T]_\beta$ is upper-triangular, we can read the eigenvalues off the diagonal.

Let $\lambda_1 = 0$ $\quad [T]_\beta - 0I = [T]_\beta \rightarrow \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$. So $v_1 = (1,0,0)$ is an eigenvector of $[T]_\beta$

$\lambda_2 = 1 \quad [T]_\beta - I = \begin{bmatrix} -1 & -1 & 0 \\ 0 & 0 & -2 \\ 0 & 0 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ So $v_2 = (1,-1,0)$ is an eigenvector of $[T]_\beta$.

$\lambda_3 = 2 \quad [T]_\beta - 2I = \begin{bmatrix} -2 & -1 & 0 \\ 0 & -1 & -2 \\ 0 & 0 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$, so $v_3 = (1,-2,1)$ is an eigenvector of $[T]_\beta$.

Then $\gamma = \{1, 1-x, 1-2x+x^2\}$, and $[T]_\gamma = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$.

6. (12 points)

(a) Find the determinant of the following matrix. Briefly justify your answer.

$$A = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 2 & 3 \\ 0 & 0 & 0 & 1 & 4 & 0 \\ 0 & 0 & 5 & 2 & 0 & -1 \\ 0 & -7 & 2 & 17 & -3 & 0 \\ 4 & 23 & 0 & 6 & 2 & 5 \end{pmatrix}$$

If we swap R_1 and R_6, R_2 and R_5, and R_3 and R_4, we get an upper triangular matrix B. The product of its diagonal entries is $(4)(-7)(5)(1)(2)(-1) = 280$.

Since we did three permutations, $\det(A) = (-1)^3 \det(B)$,

to get $\det(A) = -280$.

(b) A skew symmetric matrix is one that satisfies $A^T = -A$. If A is $n \times n$, for what values of n must $\det(A) = 0$?

$\det(-A) = (-1)^n \det(A)$. Also, $\det(A^T) = \det(A)$.

If $\det(A^T) = \det(-A)$, then $\det(A) = (-1)^n \det(A)$.

So $\det(A) = 0$ if n is odd.

If n is even, it is not necessarily the case that $\det A = 0$. Consider $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$.

7. (12 points)

(a) Suppose V is an n-dimensional vector space and $T \in \mathcal{L}(V)$. Suppose $T^2 = T_0$. Can T be invertible? Why or why not?

No. Let $A = [T]_\beta$ for some basis β for V. Since $T^2 = T_0$, $\det(A^2) = (\det(A))^2 = 0$. Hence $\det A = 0$. Since A is not invertible, T is not invertible.

(b) Suppose V is a vector space. Prove that the set of non-invertible linear operators on V is not a subspace of $\mathcal{L}(V)$.

The sum of non-invertible linear transformations can be invertible. Consider $T_1 \in \mathcal{L}(\mathbb{R}^2)$ by $T_1(a,b) = (a,0)$. Since $N(T_1) = \text{span}\{(0,1)\}$, T_1 has non-trivial nullspace, and is not invertible. Likewise $T_2 \in \mathcal{L}(\mathbb{R}^2)$ given by $T_2(a,b) = (0,b)$ is not invertible. But $T_1 + T_2$ is the identity.

So the set of non-invertible linear operators on V is not closed under addition.

8. (12 points)

Let $T : M_{2\times2}(\mathbb{R}) \to: M_{2\times2}(\mathbb{R})$ be given by $T(A) = MA$, where $M = \begin{bmatrix} 0 & 2 \\ 2 & 0 \end{bmatrix}$.

(a) Determine a basis for the T-cyclic subspace W_1 generated by $E^{11} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and one for the T-cyclic subspace W_2 generated by $E^{12} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$.

(Show your work in order to justify your conclusion. This question continues on the next page.)

W_1: $\begin{bmatrix} 0 & 2 \\ 2 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 2 & 0 \end{bmatrix}$

$\begin{bmatrix} 0 & 2 \\ 2 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 2 & 0 \end{bmatrix} = \begin{bmatrix} 4 & 0 \\ 0 & 0 \end{bmatrix} = 4E^{11}$.

basis: $\beta_1 = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 2 & 0 \end{bmatrix} \right\}$

W_2: $\begin{bmatrix} 0 & 2 \\ 2 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix}$

$\begin{bmatrix} 0 & 2 \\ 2 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 0 & 4 \\ 0 & 0 \end{bmatrix} = 4E^{12}$

basis $\beta_2 = \left\{ \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix} \right\}$

(b) Determine the characteristic polynomials of T_{W_1} and T_{W_2}.

W_1 : basis: $\beta_1 = \left\{ \underset{v_1}{\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}}, \underset{v_2}{\begin{bmatrix} 0 & 0 \\ 2 & 0 \end{bmatrix}} \right\}$

$$\left.\begin{array}{l} T_{W_1}(v_1) = v_2 \\ T_{W_1}(v_2) = 4v_1 \end{array}\right\} \quad [T_{W_1}]_{\beta_1} = \begin{bmatrix} 0 & 4 \\ 1 & 0 \end{bmatrix}$$ char poly is t^2-4

W_2: basis $\beta_2 = \left\{ \underset{v_1}{\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}}, \underset{v_2}{\begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix}} \right\}$

$$\left.\begin{array}{l} T_{W_2}(v_1) = v_2 \\ T_{W_2}(v_2) = 4v_1 \end{array}\right\} \quad [T_{W_2}]_{\beta_2} = \begin{bmatrix} 0 & 4 \\ 1 & 0 \end{bmatrix}$$ char poly is t^2-4.

(c) Use the Cayley-Hamilton theorem to show that $T^{-1} = \frac{1}{4}T$. (Hint: Note $V = W_1 \oplus W_2$.)

By Part (b) and the Cayley-Hamilton theorem, $(T^2-4I)(v)=0$ if $v \in W_1$ and $(T^2-4I)(v)=0$ if $v \in W_2$.

Since $V = W_1 \oplus W_2$, if $x \in V$, then $x = v_1 + v_2$ where $v_1 \in W_1$ and $v_2 \in W_2$. Then $(T^2-4I)(x) = (T^2-4I)(v_1) + (T^2-4I)v_2 = 0+0=0$.

Since x was arbitrarily chosen, $T^2-4I = T_0$.

Then $T^2 = 4I$, so $\frac{1}{4}T^2 = I$. This means $\frac{1}{4}T \circ T = I$. So

$T^{-1} = \frac{1}{4}T$.

Scratch paper.