To review for the midterm

- Review your notes, the text, the homework problems, and the suggested exercises in the schedule.
- Do all true false questions in the text. The exam **will** include a significant T/F section.
- Below is a list of topics for the exam and a sampling of questions from past exams. Review these also.
- VERY IMPORTANT. These are just questions from old exams for your **practice**. The questions on our exam may not be similar.

Linear transformations and matrics

- Matrix representations.
- Composition of linear transformations and matrix multiplication
- Isomorphism and Invertibility
- Computing change of coordinate matrices

Elementary Operatations and Systems of linear equations

- Know the rank of a matrix and be comfortable computing inverses using augmented matrices.
- Reduced Row Echelon form and row reduction.
- Column and row operations and multiplication by elementary matrices.
- Definition and properties of homogeneous and inhomogeneous systems and solutions.
- consistent and inconsistent systems.

Example Problems from old exams

(1) Let $T : P_2(\mathbb{R}) \to P_2(\mathbb{R})$ be defined by $T(a + bx + cx^2) = a - 3b + 5cx + (a + c)x^2$.. Let $\gamma = \{1 - x^2, x^2 + x, -5 + 4x^2\}$. Find $[T]_\gamma$. Prove that T is an isomorphism.

(2) We say that A is a submatrix of B if we have

$$B = \begin{pmatrix} * & * & * \\ * & A & * \\ * & * & * \end{pmatrix},$$

where the $''*''$ can be any matrices (of the appropriate dimensions). Prove that

$$\text{rank(A)} \leq \text{rank(B)}.$$

(3) Let

$$A = \begin{pmatrix} 1 & -2 & -1 \\ -2 & 6 & 2 \\ 0 & 1 & 3 \\ 3 & -4 & -2 \end{pmatrix}$$

Solve the linear system $Ax = 0$. Let $b = (1, 0, 4, k)^T$, and consider the system $Ax = b$. Find a real number k such that the system is inconsistent, or explain why this is not possible. Find a real number k such that the system has infinitely-many solutions, or explain why this is not possible. Find a real number k such that the system has exactly one solution, or explain why this is not possible.

(4) Let the reduced row echelon form of A be

$$\begin{pmatrix} 1 & -3 & 0 & 4 & 0 & 5 \\ 0 & 0 & 1 & 3 & 0 & 2 \\ 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Determined A if the first, third, and sixth colums of A are

$$\begin{pmatrix} 1 \\ -2 \\ -1 \\ 3 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \\ 2 \\ -4 \end{pmatrix}, \begin{pmatrix} 3 \\ -9 \\ 2 \\ 5 \end{pmatrix}$$

(5) n Let a and b be two distinct real numbers, and consider T from $P_1(\mathbb{R})$ to $\mathbb{R}^2$ defined by $T(f(x)) = (f(a), f(b))$. Show that T is linear and invertible. Given c and d, compute $T^{-1}(c, d)$.

(6) Let V and W be vector spaces. Prove that V and W are isomorphic if and only if there are bases β and γ for V and W respectively and a transformation $T : V \to W$ such that $[T]_\beta^\gamma$ is the identity matrix.

(7) Let $B = \begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix}$ and define the mapping $T : M_{2\times 2}(\mathbb{R}) \to \mathbb{R}$ by $T(A) = trace(AB)$. Show that T is linear and compute the rank of T. Show that $\begin{pmatrix} 1 & -1 \\ 2 & -1 \end{pmatrix}$ is in the nullspace of T. Find a basis for $N(T)$ which contains this matrix.

(8) in $\mathbb{R}^2$, Let $\beta = \{(1, 2), (3, 4)\}$ and $\beta' = \{(2, 4), (4, 6)\}$. Find the change coordinate matrix taking β' coordinates to β coordinates.

(9) Find all solutions to the system

$$\begin{aligned} x_1 + 2x_2 + 5x_3 &= 1 \\ x_1 - x_2 - x_3 &= 2. \end{aligned}$$

Write down a product of elementary matrices that transforms the matrix of the system to its reduced row echelon form

(10) How many solutions can a homogeneous system of linear equations have? Give an example of a system of two equations in two variables for each case. Explain your examples briefly. You do not have to find the solutions of the systems.

(1) Let $T : P_2(\mathbb{R}) \to P_2(\mathbb{R})$ be defined by $T(a + bx + cx^2) = a - 3b + 5cx + (a + c)x^2$.. Let $\gamma = \{1 - x^2, x^2 + x, -5 + 4x^2\}$. Find $[T]_\gamma$. Prove that T is an isomorphism.

$$[T]_\beta = \begin{bmatrix} 1 & -3 & 0 \\ 0 & 0 & 5 \\ 1 & 0 & 1 \end{bmatrix} \qquad [I]_\gamma^\beta = \begin{bmatrix} 1 & 0 & -5 \\ 0 & 1 & 0 \\ -1 & 1 & 4 \end{bmatrix} = Q$$

Q^{-1}:

$$\left[\begin{array}{ccc|ccc} 1 & 0 & -5 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ -1 & 1 & 4 & 0 & 0 & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 0 & -5 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & -1 & 1 & 0 & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 0 & -5 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & +1 & -1 & +1 & -1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -4 & 5 & -5 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & -1 & 1 & -1 \end{array}\right]$$

$$[T]_\gamma = \begin{bmatrix} -4 & 5 & -5 \\ 0 & 1 & 0 \\ -1 & 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & -3 & 0 \\ 0 & 0 & 5 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & -5 \\ 0 & 1 & 0 \\ -1 & 1 & 4 \end{bmatrix} = \begin{bmatrix} -29 & 32 & 125 \\ -5 & 5 & 20 \\ -6 & 7 & 26 \end{bmatrix}$$

(2) We say that A is a submatrix of B if we have

$$B = \begin{pmatrix} * & * & * \\ * & A & * \\ * & * & * \end{pmatrix},$$

where the "$*$" can be any matrices (of the appropriate dimensions). Prove that

$$\text{rank(A)} \leq \text{rank(B)}.$$

Suppose A has rank r. Suppose $\alpha_{j1}, \alpha_{j2}, \dots, \alpha_{jr}$ are independent columns in A.

Let $\beta_{k1}, \beta_{k2}, \dots, \beta_{kr}$ be columns in B of the form

$$\beta_{k\ell} = \begin{bmatrix} u_\ell \\ \alpha_{j\ell} \\ v_\ell \end{bmatrix}$$ where u_ℓ and v_ℓ are column vectors

of the appropriate dimension. Set

$c_1 \beta_{k1} + c_2 \beta_{k2} + \dots + c_r \beta_{kr} = 0$. Then it must also be

true that $c_1 \alpha_{j1} + c_2 \alpha_{j2} + \dots + c_r \alpha_{jr} = 0$. Since the α_{ji} are independent, $c_1 = c_2 = \dots = c_r = 0$.

So the β_{ki} are independent as well. Hence $\text{rank}(B) \geq r$.

(3) Let

$$A = \begin{pmatrix} 1 & -2 & -1 \\ -2 & 6 & 2 \\ 0 & 1 & 3 \\ 3 & -4 & -2 \end{pmatrix}$$

Solve the linear system $Ax = 0$. Let $b = (1, 0, 4, k)^T$, and consider the system $Ax = b$. Find a real number k such that the system is inconsistent, or explain why this is not possible. Find a real number k such that the system has infinitely-many solutions, or explain why this is not possible. Find a real number k such that the system has exactly one solution, or explain why this is not possible.

$$\left(\begin{array}{ccc|c} 1 & -2 & -1 & 1 \\ -2 & 6 & 2 & 0 \\ 0 & 1 & 3 & 4 \\ 3 & -4 & -2 & k \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & -2 & -1 & 1 \\ 0 & 2 & 0 & 2 \\ 0 & 1 & 3 & 4 \\ 0 & 2 & 1 & k-3 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & -2 & -1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 3 & 4 \\ 0 & 2 & 1 & k-3 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 0 & -1 & 3 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 3 & 3 \\ 0 & 0 & 1 & k-5 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 0 & 0 & k-2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & k-8 \end{array}\right)$$

→ The system is inconsistent if $k \neq 8$.

→ If $k=8$, we get a unique solution, as we have shown that $\text{rank}(A) = 3$, so $\text{nullity}(A) = 0$. The the solution set for $Ax=b$ is $K = \{s_0\} + K_H$ for some s_0.

$$= \{s_0\} + \{(0,0,0)\}.$$

→ There is no possible k that gives an infinite solution set.

4) Let the reduced row echelon form of A be

$$\begin{pmatrix} 1 & -3 & 0 & 4 & 0 & 5 \\ 0 & 0 & 1 & 3 & 0 & 2 \\ 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Determined A if the first, third, and ~~sixth~~ 5th colums of A are

$$\begin{pmatrix} 1 \\ -2 \\ -1 \\ 3 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \\ 2 \\ -4 \end{pmatrix}, \begin{pmatrix} 3 \\ -9 \\ 2 \\ 5 \end{pmatrix}$$

$$\left.\begin{aligned} c_2 &= -3c_1 \\ c_4 &= 4c_1 + 3c_3 \\ c_6 &= 5c_1 + 2c_3 - c_5 \end{aligned}\right\}$$

$$c_2 = \begin{pmatrix} -3 \\ 6 \\ 3 \\ -9 \end{pmatrix} \qquad c_4 = 4\begin{pmatrix} 1 \\ -2 \\ -1 \\ 3 \end{pmatrix} + 3\begin{pmatrix} -1 \\ 1 \\ 2 \\ -4 \end{pmatrix} = \begin{pmatrix} 1 \\ -5 \\ 2 \\ 0 \end{pmatrix}$$

$$c_6 = 5\begin{pmatrix} 1 \\ -2 \\ -1 \\ 3 \end{pmatrix} + 2\begin{pmatrix} -1 \\ 1 \\ 2 \\ -4 \end{pmatrix} - \begin{pmatrix} 3 \\ -9 \\ 2 \\ 5 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ -3 \\ 2 \end{pmatrix}$$

$$A = \begin{pmatrix} 1 & -3 & -1 & 1 & 3 & 0 \\ -2 & 6 & 1 & -5 & -9 & 1 \\ -1 & 3 & 2 & 2 & 2 & -3 \\ 3 & -9 & -4 & 0 & 5 & 2 \end{pmatrix}$$

$\backslash\ 3\ /\quad \backslash -4 /\quad \backslash\ 5\ /$

(5) n Let a and b be two distinct real numbers, and consider T from $P_1(\mathbb{R})$ to $\mathbb{R}^2$ defined by $T(f(x)) = (f(a), f(b))$. Show that T is linear and invertible. Given c and d, compute $T^{-1}(c, d)$.

$$T(a_1 + b_1 x + \lambda(a_2 + b_2 x)) = T(a_1 + \lambda a_2 + x(b_1 + \lambda b_2))$$
$$= (a_1 + \lambda a_2 + (b_1 + \lambda b_2)a,\ a_1 + \lambda a_2 + (b_1 + \lambda b_2)b)$$
$$= (a_1 + b_1 a) + \lambda a_2 + \lambda b_2 a,\ a_1 + b_1 b + \lambda a_2 + \lambda b_2 b)$$
$$= (a_1 + b_1 a,\ a_1 + b_1 b) + \lambda(a_2 + b_2 a,\ a_2 + b_2 b)$$
$$= T(a_1 + b_1 x) + \lambda T(a_2 + b_2 x).$$

Let β, γ be the standard ordered bases for P_1 & $\mathbb{R}^2$ respectively.

$$[T]_\beta^\gamma = \begin{bmatrix} 1 & a \\ 1 & b \end{bmatrix} \quad ([T]_\beta^\gamma)^{-1} = [T^{-1}]_\gamma^\beta = \frac{1}{b-a}\begin{bmatrix} b & -a \\ -1 & 1 \end{bmatrix}$$

$$[T^{-1}]_\gamma^\beta \begin{bmatrix} c \\ d \end{bmatrix} = \frac{1}{b-a}\begin{bmatrix} b & -a \\ -1 & 1 \end{bmatrix}\begin{bmatrix} c \\ d \end{bmatrix} = \frac{1}{b-a}\begin{bmatrix} bc-ad \\ d-c \end{bmatrix}.$$

So $T^{-1}(c,d) = \dfrac{bc-ad}{b-a} + \left(\dfrac{d-c}{b-a}\right)x$.

compute $T^{-1}(c, d)$.

(6) Let V and W be vector spaces. Prove that V and W are isomorphic if and only if there are bases β and γ for V and W respectively and a transformation $T : V \to W$ such that $[T]_\beta^\gamma$ is the identity matrix.

Suppose there exists such a T. Then $[T]_\beta^\gamma = I_n$, so T is invertible. So $T: V \to W$ is an isomorphism.

Suppose V, W are isomorphic. Let $\beta = \{b_1, b_2, \ldots, b_n\}$ be an ordered basis for V. Let $\gamma = \{g_1, g_2, \ldots, g_n\}$ be an ordered basis for W. Since V is isomorphic to W, they have the same dimension.

Define $T: V \to W$ by $T(b_i) = g_i$.

Then $$[T]_\beta^\gamma = [[T(b_1)]_\gamma \mid \cdots \mid [T(b_n)]_\gamma]$$
$$= [e_1 \mid \cdots \mid e_n] = I_n.$$

(7) Let $B = \begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix}$ and define the mapping $T : M_{2\times 2}(\mathbb{R}) \to \mathbb{R}$ by $T(A) = trace(AB)$.

Show that T is linear and compute the rank of T. Show that $\begin{pmatrix} 1 & -1 \\ 2 & -1 \end{pmatrix}$ is in the nullspace of T. Find a basis for $N(T)$ which contains this matrix.

$Tr(AB) = Tr\left[\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix}\right] = Tr\begin{pmatrix} -a & a+b \\ -c & c+d \end{pmatrix} = c+d-a.$

$T(A_1+\lambda A_2) = Tr\,(A_1 + \lambda A_2)(B) = Tr[A_1 B + \lambda A_2 B]$

$= Tr(A_1 B) + \lambda Tr(A_2 B) = T(A_1) + \lambda T(A_2).$

rank$(T) = 1$. (It's either 0 or 1, as $T\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \neq 0$, so it's 1.)

$\begin{pmatrix} 1 & -1 \\ 2 & -1 \end{pmatrix} \in N(T)$, because $c+d-a = 2-1-1 = 0$.

$K_H = \left\{ \begin{pmatrix} c+d & b \\ c & d \end{pmatrix} \right\} = span\left\{ \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right\}$

Set $\begin{pmatrix} 1 & -1 \\ 2 & -1 \end{pmatrix}$ as the first column with the other columns from here.

$$\begin{pmatrix} 1 & 1 & 0 & 1 \\ -1 & 0 & 1 & 0 \\ 2 & 1 & 0 & 1 \\ -1 & 0 & 0 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

so the first 3 cols are independent. the basis is

$\left\{ \begin{pmatrix} 1 & -1 \\ 2 & -1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \right\}.$

(8) in $\mathbb{R}^2$, Let $\beta = \{(1,2),(3,4)\}$ and $\beta' = \{(2,4),(4,6)\}$. Find the change coordinate matrix taking β' coordinates to β coordinates.

$$\left[\begin{array}{cc|cc} 1 & 3 & 2 & 4 \\ 2 & 4 & 4 & 6 \end{array}\right] \to \left[\begin{array}{cc|cc} 1 & 3 & 2 & 4 \\ 0 & -2 & 0 & -2 \end{array}\right] \to \left[\begin{array}{cc|cc} 1 & 3 & 2 & 4 \\ 0 & 1 & 0 & 1 \end{array}\right] \to \left[\begin{array}{cc|cc} 1 & 0 & 2 & 1 \\ 0 & 1 & 0 & 1 \end{array}\right]$$

$[I]_{\beta'}^{\beta} = \begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix}$

check: $v = a(2,4) + b(4,6) \leftarrow [v]_{\beta'} = \begin{bmatrix} a \\ b \end{bmatrix}$

$= (2a+4b,\ 4a+6b)$

$\begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 2a+b \\ b \end{bmatrix} = [v]_\beta$

$(2a+b)(1,2) + b(3,4)$

$= (2a+b+3b,\ 4a+2b+4b)$

$= (2a+4b,\ 4a+6b)$

alt: $[I]_{\beta'}^{\gamma} = \begin{bmatrix} 2 & 4 \\ 4 & 6 \end{bmatrix}$

$[I]_{\beta}^{\gamma} = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}$

Where γ is the standard basis.

then $[I]_{\beta'}^{\beta} = [I]_{\gamma}^{\beta'} [I]_{\beta}^{\gamma}$

matrix taking β-coordinates to β'-coordinates.

(9) Find all solutions to the system

$$x_1 + 2x_2 + 5x_3 = 1$$
$$x_1 - x_2 - x_3 = 2.$$

Write down a product of elementary matrices that transforms the matrix of the system to its reduced row echelon form

$$(A|b) = \left[\begin{array}{ccc|c} 1 & 2 & 5 & 1 \\ 1 & -1 & -1 & 2 \end{array}\right] \xrightarrow{R_2 - R_1} \left[\begin{array}{ccc|c} 1 & 2 & 5 & 1 \\ 0 & -3 & -6 & 1 \end{array}\right]$$

$$\xrightarrow{-\frac{1}{3}R_2} \left[\begin{array}{ccc|c} 1 & 2 & 5 & 1 \\ 0 & 1 & 2 & -\frac{1}{3} \end{array}\right] \xrightarrow{R_1 - 2R_2} \left[\begin{array}{ccc|c} 1 & 0 & 1 & \frac{5}{3} \\ 0 & 1 & 2 & -\frac{1}{3} \end{array}\right]$$

Let $x_3 = t$

Solution vector: $$\begin{pmatrix} \frac{5}{3} - t \\ -\frac{1}{3} - 2t \\ t \end{pmatrix} = \left\{ \begin{pmatrix} \frac{5}{3} \\ -\frac{1}{3} \\ 0 \end{pmatrix} + t\begin{pmatrix} -1 \\ -2 \\ 1 \end{pmatrix} \;\middle|\; t \in F \right\}.$$

$$E_3 = R_2 - R_1 = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} \quad E_2 = -\frac{1}{3}R_2 \begin{bmatrix} 1 & 0 \\ 0 & -\frac{1}{3} \end{bmatrix} \quad E_1 = R_1 - 2R_2 = \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix}$$

$$P = E_1E_2E_3 = \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & -\frac{1}{3} \end{bmatrix}\begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ \frac{1}{3} & -\frac{1}{3} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{3} & \frac{2}{3} \\ \frac{1}{3} & -\frac{1}{3} \end{bmatrix}$$

Check $$\begin{bmatrix} \frac{1}{3} & \frac{2}{3} \\ \frac{1}{3} & -\frac{1}{3} \end{bmatrix}\left[\begin{array}{ccc|c} 1 & 2 & 5 & 1 \\ 1 & -1 & -1 & 2 \end{array}\right].$$

$$= \left[\begin{array}{ccc|c} 1 & 0 & 1 & \frac{5}{3} \\ 0 & 1 & 2 & -\frac{1}{3} \end{array}\right]$$

(10) How many solutions can a homogeneous system of linear equations have? Give an example of a system of two equations in two variables for each case. Explain your examples briefly. You do not have to find the solutions of the systems.

Homogeneous systems are always consistent. They can have either the unique solution 0 or ∞-many solutions.

Unique: $x_1 + x_2 = 0$, $x_2 = 0$

∞-many: $x_1 + x_2 = 0$, $2x_1 + 2x_2 = 0$