MTH 235 Spring 2023

Midterm 1, February 16, 2023

NAME (please print legibly): Key ______________________________

Your University ID Number: ______________________________

Please circle your instructor's name: **Madhu** **Kleene**

- No notes or electronic devices are permitted during the exam.
- Full justification is required on all questions except the True/False. In particular, if you provide a counter-example, you must explain why your counter-example is appropriate.
- Please initial to indicate that you have read and understood these instructions.________

PLEASE COPY THE HONOR PLEDGE AND SIGN. (Cursive is not required).
I affirm that I will not give or receive any unauthorized help on this exam, and all work will be my own.

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YOUR SIGNATURE:______________________________

1. (20 points) Let $V = \{(a_1, a_2) \mid a_i \in \mathbb{R}\}$. For $\mathbf{u} = (u_1, u_2), \mathbf{v} = (v_1, v_2) \in V$ and $\lambda \in \mathbb{R}$ we define vector addition ($\boxplus$) and scalar multiplication ($\boxdot$) below. In each case determine why V is not a vector space over $\mathbb{R}$ with those operations and justify your answer by providing a specific counterexample.

(a) $\mathbf{u} \boxplus \mathbf{v} = (u_1 - v_1, u_2 - v_2)$ and $\lambda \boxdot \mathbf{u} = (\lambda u_1, \lambda u_2)$.

This addition is not commutative.

Let $u = (1,2)$ and $v = (3,4)$.

Then $u \boxplus v = (-2,-2)$ and $v \boxplus u = (2,2)$.

(b) $\mathbf{u} \boxplus \mathbf{v} = (u_1 + v_1, u_2 + v_2)$ and $\lambda \boxdot \mathbf{u} = \begin{cases} \mathbf{0} & \text{if } \lambda = 0 \\ (\lambda u_1, \frac{u_2}{\lambda}) & \text{if } \lambda \neq 0 \end{cases}$

This multiplication fails VS8

Let $u = (1,2)$ and $\lambda_1 = 3$ and $\lambda_2 = 4$

$(\lambda_1 + \lambda_2) \boxdot u = 7 \boxdot (1,2) = (7, \frac{2}{7})$.

$(\lambda_1 \boxdot u) \boxplus (\lambda_2 \boxdot u) = 3 \boxdot (1,2) \boxplus 4 \boxdot (1,2)$

$= (3, \frac{2}{3}) \boxplus (4, \frac{1}{2})$

$= (-1, \frac{1}{6})$.

2. (20 points) Prove that $\mathbb{R}^3$ is the sum of the subspaces $W_1 = \{(x, y, 0) | \, x, y, \in \mathbb{R}\}$ and $W_2 = \{(0, y, z) | \, y, z \in \mathbb{R}\}$. Also, prove that the sum is not a direct sum.

Let $v \in \mathbb{R}^3$. Then $v = (a, b, c)$ for some $a, b, c \in \mathbb{R}$.

So $v = (a, b, 0) + (0, 0, c) \in W_1 + W_2$.

Then $\mathbb{R}^3 \subseteq W_1 + W_2$.

Let $v \in W_1 + W_2$. Then $v = (a, b, 0) + (0, c, d)$ for some $a, b, c, d \in \mathbb{R}$.

Since $v = (a, b+c, d) \in \mathbb{R}^3$, $W_1 + W_2 \subseteq \mathbb{R}^3$, and we have equality.

(Note: this second direction was proved in homework, so not necessary to show here.)

Since $(0, 1, 0) \in W_1 \cap W_2$, the sum is not direct.

3. (20 points)

Let V be a vector space and W_1,W_2 subspaces of V. Suppose $\dim(W_1) = m$, $\dim(W_2) = n$, and $m > n$.

(a) Show that $\dim(W_1 \cap W_2) \leq n$.

Since $W_1 \cap W_2 \subseteq W_2$, $\dim(W_1 \cap W_2) \leq \dim(W_2) = n$.

(b) Suppose that $W_1 \cup W_2$ is a subspace. Are you able to determine its dimension? Why or why not?

If $W_1 \cup W_2$ is a subspace, either $W_1 \subseteq W_2$ or $W_2 \subseteq W_1$. Since $\dim(W_1) > \dim(W_2)$, $W_2 \subseteq W_1$. So $W_1 \cup W_2 = W_1$, which has dimension m.

4. (20 points)

Find a basis for the null space and range of the linear transformation $T : M_{2\times 2}(\mathbb{R}) \to P_2(\mathbb{R})$ given by

$$T\begin{pmatrix} a & b \\ c & d \end{pmatrix} = (a+c+2d)x^2 + (2a+b+c+4d)x + (3b-3c)$$

If $T(M) = (0,0,0)$, then
$$a+c+2d=0 \quad ①$$
$$2a+b+c+4d=0 \quad ②$$
$$3b-3c=0. \quad ③$$

Since ③ shows $b=c$, ② reduces to twice equation ①.
Then we only need to satisfy $b=c$ and $a=-c-2d$.

Then $N(T) = \left\{ \begin{pmatrix} -c-2d & c \\ c & d \end{pmatrix} \middle| \, c,d \in \mathbb{R} \right\} = \text{span}\left\{ \begin{pmatrix} -1 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} -2 & 0 \\ 0 & 1 \end{pmatrix} \right\}$

This spanning set is independent:
Setting $\lambda_1 \begin{pmatrix} -1 & 1 \\ 1 & 0 \end{pmatrix} + \lambda_2 \begin{pmatrix} -2 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} -\lambda_1 - 2\lambda_2 & \lambda_1 \\ \lambda_1 & \lambda_2 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$,

we get $\lambda_1 = \lambda_2 = 0$.

Then $\beta = \left\{ \begin{pmatrix} -1 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} -2 & 0 \\ 0 & 1 \end{pmatrix} \right\}$ is a basis for $N(T)$.

To find $R(T)$, we apply T to the basis for $M_{2\times 2}(\mathbb{R})$.

$T(E^{11}) = x^2 + x$
$T(E^{12}) = x - 3$

Since x^2+x and $x-3$ are not scalar multiples of each other, we can stop there. We can see that $\{x^2+x, x-3\}$ is independent and can be extended to a basis for $R(T)$. But from the dimension theorem, we know $\dim(R(T)) = 2$.

So $\gamma = \{x^2+x, x-3\}$ is a basis for $R(T)$

5. (20 points) Choose True or False for each question below. You do not have to justify your answers, and partial credit will not be offered.

1. The zero vector is a linear combination of any non-empty subset of a vector space V.

   ☒ TRUE ☐ FALSE

2. Every subspace of a finite-dimensional vector space is finite-dimensional.

   ☒ TRUE ☐ FALSE

3. Suppose $T : \mathbb{R}^5 \to \mathbb{R}^6$ is linear and $\dim(N(T)) = 1$. Then T is one-to-one.

   ☐ TRUE ☒ FALSE

4. Suppose $T : \mathbb{R}^5 \to \mathbb{R}^4$ is linear and $\dim(N(T)) = 1$. Then T is onto.

   ☒ TRUE ☐ FALSE

5. If $S \subset V$ is a subset, then we can always extend S to a basis for V.

   ☐ TRUE ☒ FALSE

6. Let S be a subset of a vector space V, and suppose $\dim(V) = n$. Suppose S generates V. Then any linearly independent subset of S has at most n vectors, and S has at least n vectors.

☒ TRUE ☐ FALSE

7. If V is a vector space of finite dimension n, then V has exactly one subspace of dimension 0 and one of dimension n.

☒ TRUE ☐ FALSE

8. Let $T : P_2(\mathbb{R}) \to P_3(\mathbb{R})$ be given by $T(f) = \int\limits_0^x f(t)dt$. Then T is one-to-one.

☒ TRUE ☐ FALSE

9. If $\lambda \in F$ and $x, y \in V$ and $\lambda x = \lambda y$, then $x = y$.

☐ TRUE ☒ FALSE

10. Let $v_1, v_2, v_3, v_4, v_5 \in \mathbb{R}^5$. If both $\{v_1, v_2, v_3\}$ and $\{v_4, v_5\}$ are linearly independent sets, then $\{v_1, v_2, v_3, v_4, v_5\}$ is a linearly independent set.

☐ TRUE ☒ FALSE

Use this page for scratch work.