MTH 235 Spring 2023

Midterm 1, February 16, 2023

NAME (please print legibly): ____
Your University ID Number: ____
Your University email ____

Please circle your instructor's name: **Madhu** **Kleene**

- No notes or electronic devices are permitted during the exam.
- Full justification is required on all questions except the True/False. In particular, if you provide a counter-example, you must explain why your counter-example is appropriate.
- Please initial to indicate that you have read and understood these instructions.________

PLEASE COPY THE HONOR PLEDGE AND SIGN. (Cursive is not required).
I affirm that I will not give or receive any unauthorized help on this exam, and all work will be my own.

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YOUR SIGNATURE:____

QUESTION	VALUE	SCORE
1	20	
2	20	
3	20	
4	20	
5	20	
TOTAL	100	

1. (20 points) Let $V = \{(a_1, a_2) \mid a_i \in \mathbb{R}\}$. For $\mathbf{u} = (u_1, u_2), \mathbf{v} = (v_1, v_2) \in V$ and $\lambda \in \mathbb{R}$ we define vector addition ($\boxplus$) and scalar multiplication ($\boxdot$) below. In each case determine why V is not a vector space over $\mathbb{R}$ with those operations and justify your answer by providing a specific counterexample.

(a) $\mathbf{u} \boxplus \mathbf{v} = (u_1 - v_1, u_2 - v_2)$ and $\lambda \boxdot \mathbf{u} = (\lambda u_1, \lambda u_2)$.

(b) $\mathbf{u} \boxplus \mathbf{v} = (u_1 + v_1, u_2 + v_2)$ and $\lambda \boxdot \mathbf{u} = \begin{cases} \mathbf{0} & \text{if } \lambda = 0 \\ (\lambda u_1, \dfrac{u_2}{\lambda}) & \text{if } \lambda \neq 0 \end{cases}$

2. (20 points) Prove that $\mathbb{R}^3$ is the sum of the subspaces $W_1 = \{(x, y, 0)|\, x, y, \in \mathbb{R}\}$ and $W_2 = \{(0, y, z)|\, y, z \in \mathbb{R}\}$. Also, prove that the sum is not a direct sum.

3. (20 points)

Let V be a vector space and W_1,W_2 subspaces of V. Suppose $\dim(W_1) = m$, $\dim(W_2) = n$, and $m > n$.

(a) Show that $\dim(W_1 \cap W_2) \leq n$.

(b) Suppose that $W_1 \cup W_2$ is a subspace. Are you able to determine its dimension? Why or why not?

4. (20 points)

Find a basis for the null space and range of the linear transformation $T : M_{2\times 2}(\mathbb{R}) \to P_2(\mathbb{R})$ given by

$$T\begin{pmatrix} a & b \\ c & d \end{pmatrix} = (a + c + 2d)x^2 + (2a + b + c + 4d)x + (3b - 3c)$$

5. (20 points) Choose True or False for each question below. You do not have to justify your answers, and partial credit will not be offered.

1. The zero vector is a linear combination of any non-empty subset of a vector space V.

   ☐ TRUE ☐ FALSE

2. Every subspace of a finite-dimensional vector space is finite-dimensional.

   ☐ TRUE ☐ FALSE

3. Suppose $T : \mathbb{R}^5 \to \mathbb{R}^6$ is linear and $\dim(N(T)) = 1$. Then T is one-to-one.

   ☐ TRUE ☐ FALSE

4. Suppose $T : \mathbb{R}^5 \to \mathbb{R}^4$ is linear and $\dim(N(T)) = 1$. Then T is onto.

   ☐ TRUE ☐ FALSE

5. If $S \subset V$ is a subset, then we can always extend S to a basis for V.

   ☐ TRUE ☐ FALSE

6. Let S be a subset of a vector space V, and suppose $\dim(V) = n$. Suppose S generates V. Then any linearly independent subset of S has at most n vectors, and S has at least n vectors.

☐ TRUE ☐ FALSE

7. If V is a vector space of finite dimension n, then V has exactly one subspace of dimension 0 and one of dimension n.

☐ TRUE ☐ FALSE

8. Let $T : P_2(\mathbb{R}) \to P_3(\mathbb{R})$ be given by $T(f) = \int_0^x f(t)dt$. Then T is one-to-one.

☐ TRUE ☐ FALSE

9. If $\lambda \in F$ and $x, y \in V$ and $\lambda x = \lambda y$, then $x = y$.

☐ TRUE ☐ FALSE

10. Let $v_1, v_2, v_3, v_4, v_5 \in \mathbb{R}^5$. If both $\{v_1, v_2, v_3\}$ and $\{v_4, v_5\}$ are linearly independent sets, then $\{v_1, v_2, v_3, v_4, v_5\}$ is a linearly independent set.

☐ TRUE ☐ FALSE

Use this page for scratch work.