MATH 235

Final

May 4, 2015

Course ID number: ____________

Circle your professor's name: **Mueller** **Petridis**

- **No calculators are allowed on this exam. The use of cell phones, computers, tablets, and so on is not permitted.**
- **Two sheets of notes are allowed with writing on both sides.**
- **You must explain your answers, and provide a proof if the question asks for one.**

QUESTION	VALUE	SCORE
1	10	
2	10	
3	10	
4	10	
5	10	
6	10	
7	10	
8	10	
9	10	
10	10	
TOTAL	100	

1. (10 points)

Fix positive integers m, n and consider the vector space $\mathbf{V}$ of all $m \times n$ matrices with entries in the real numbers $\mathbf{R}$.

(a) Find the dimension of $\mathbf{V}$ and prove your answer. Please carry out all the steps of your proof.

(b) Let $\mathbf{P}$ be the subset of $\mathbf{V}$ consisting of $m \times n$ matrices each of whose row sum is 1. Prove or disprove: $\mathbf{P}$ is a subspace of $\mathbf{V}$.

(c) Assume $m \geq 2$ and $n \geq 2$. Find a subspace of $\mathbf{V}$ of dimension 2. Please explain your answer, but you don't have to give a proof.

2. (10 points)

$P_2(\mathbf{R})$ is the real vector space of real polynomials of degree at most 2. Let W be the following subset of $P_2(\mathbf{R})$:

$$W = \{f \in P_2(\mathbf{R}) \mid f(2) = f(1)\}.$$

(a) Prove that W is a vector subspace of $P_2(\mathbf{R})$.

(b) Write down a basis for W. You do not need to prove that the set given is a basis, though justification of how you found it must be given.

(c) W is isomorphic to $\mathbf{R}^d$ for what value of d? Justify your answer.

3. (10 points)

Let $\mathbf{V}$ denote the linear span of the following functions from $\mathbf{R}$ to $\mathbf{R}$: $e^{-2x}, 1, e^{2x}$. Also suppose that these functions form an ordered basis β for $\mathbf{V}$. Let $T : \mathbf{V} \to \mathbf{V}$ be the linear transformation defined by $(Tf)(x) = f(-x)$, and let $D : \mathbf{V} \to \mathbf{V}$ be the linear transformation defined by $(Df)(x) = \frac{df(x)}{dx}$.

For the following questions, you must show your calculations, but you need not give a proof.

(a) Find the matrix $[T]_\beta$.

(b) Find the matrix $[D]_\beta$.

(c) Find the matrix $[TD]_\beta$.

4. (10 points)

Let $T : \mathbf{R}^2 \to \mathbf{R}^2$ be linear and suppose $T^2 \neq 0$, where $T^2 = T \circ T$ and 0 denotes the zero map.

(a) Show that $1 \leq \text{rank}(T^2) \leq \text{rank}(T)$.

(b) By considering the possible values of $\text{rank}(T)$ separately, deduce that $R(T) = R(T^2)$, where, say, $R(T)$ is the range of T.

5. (10 points)

Use elementary row and/or column operations to find the determinant of

$$A = \begin{pmatrix} 1 & 1 & 2 & 0 \\ 1 & 0 & 1 & 3 \\ 2 & 1 & 1 & 2 \\ 0 & 2 & 1 & 3 \end{pmatrix}.$$

You must use the method of row and column operations to get any credit for this problem. It's also the easiest way.

6. (10 points)

Let $T : \mathbf{R}^2 \to \mathbf{R}^2$ be a linear map and β the following basis for $\mathbf{R}^2$:

$$\beta = \left\{ \begin{pmatrix} 1 \\ 3 \end{pmatrix}, \begin{pmatrix} 2 \\ 4 \end{pmatrix} \right\}.$$

Suppose that T is represented by the following matrix A in β:

$$A := [T]_\beta = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}.$$

(a) Find the nullity of T showing all your work.

(b) Find the matrix representing T in the standard basis for $\mathbf{R}^2$ showing all your work.

7. (10 points)

$P_1(\mathbf{R})$ is the real vector space of real polynomials of degree at most 1. Consider the linear maps $T : P_1(\mathbf{R}) \to P_1(\mathbf{R})$ and $S : P_1(\mathbf{R}) \to P_1(\mathbf{R})$ given by

$$T(p)(t) = 2p(t) + p'(t) \text{ and } S(p)(t) = p(t) + (t+1)p'(t).$$

Answer the following questions by performing calculations. You need not give a proof.

(a) Find the eigenvalues of T and S.

(b) Find the corresponding eigenvectors for T and S.

(c) Which of the linear transformations T, S are diagonalizable?

8. (10 points)

Let A and B be real $n \times n$ square matrices.

(a) Suppose that AB is not invertible. Is it true that at least one of A and B is not invertible? Provide a proof or counter example.

(b) Suppose that A has at most $n-1$ nonzero entries, that is at most $n-1$ of the $A_{ij} \neq 0$. Is it true that $\det(A) = 0$? Provide a proof or counter example.

(c) Suppose that A and B commute, that is $AB = BA$. Is it true that $\det(A^2 - B^2) = \det(A - B)\det(A + B)$? Provide a proof or counter example.

(d) Suppose that $A^k = I_n$ for some positive integer $k > 0$. What are the possible values of $\det(A)$? Justify your answer.

9. (10 points)

Recall that we say an $n \times n$ matrix A over the complex numbers is self-adjoint if $A^* = A$, where A^* is the complex conjugate of the transpose of A.

We call an $n \times n$ matrix A over the complex numbers a *Mueller-Petridis matrix* if $A^* = 3A$.

(a) Give an example of a 2×2 Mueller-Petridis matrix.

(b) Give a complete list of $n \times n$ Mueller-Petridis matrices, and prove your answer.

(c) Suppose A is an $n \times n$ matrix over the complex numbers, and $A^* = \lambda A$ for some scalar λ. What are the possible values of λ, and how does λ depend on the matrix A?

10. (10 points)

Consider the following basis for $\mathbf{R}^4$.

$$\beta = \left\{ \mathbf{w}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \mathbf{w}_2 = \begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \end{pmatrix}, \mathbf{w}_3 = \begin{pmatrix} 1 \\ -1 \\ 1 \\ 0 \end{pmatrix}, \mathbf{w}_4 = \begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix} \right\}.$$

(a) Apply the Gram-Schmidt orthonormalisation process to obtain an orthonormal basis $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$. Show all your work.

(b) Find an orthonormal basis for the orthogonal complement to

$$\text{span}\left(\left\{ \mathbf{w}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \mathbf{w}_2 = \begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \end{pmatrix} \right\} \right).$$