Math 235: Linear Algebra

Final Exam

May 5, 2014

NAME (please print legibly): ______________________________

Student ID Number: ______________________________

<u>CIRCLE YOUR INSTRUCTOR</u>: **Fatima Mahmood** **Geordie Richards**

- **Read all instructions and all problems carefully.**
- **This is a closed-book and closed-notes exam. No paper of your own is allowed.**
- **No calculators, cell phones, or other electronic devices are allowed during the exam.**
- **Show your work and justify your answers, unless indicated otherwise. You may not receive credit for a correct answer given with insufficient or incorrect reasoning.**
- **You have 3 hours to work on this exam.**
- **By taking this exam, you are acknowledging that the following is prohibited by the College's Honesty Policy: Obtaining an examination prior to its administration. Using unauthorized aid during an examination or having such aid visible to you during an examination. Knowingly assisting someone else during an examination or not keeping your work adequately protected from copying by another.**

QUESTION	VALUE	SCORE
1	10	
2	8	
3	10	
4	10	
5	12	
6	10	
7	10	
8	10	
9	10	
10	10	
TOTAL	100	

1. (10 points) Prove (from the definitions) or disprove (using a counterexample) each of the following statements.

(a) $W = \{(x, y, z) : x = 3z \text{ and } z = y^2\}$ is a subspace of $\mathbb{R}^3$.

(b) $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ by $T(x, y) = (y, -x)$ is a linear transformation.

2. (8 points) Let n be a positive odd integer. Let 0_n and I_n denote the $n \times n$ zero matrix and identity matrix, respectively. Consider the following matrix:

$$A = \begin{pmatrix} 0_n & 3I_n \\ -2I_n & 0_n \end{pmatrix}$$

(a) Find $\det(A)$. You may use n in your answer. Provide your reasoning.

(b) Is $\det(A)$ positive or negative?

3. (10 points) Let $\mathcal{F}$ denote the vector space of functions $f : \mathbb{R} \to \mathbb{R}$ over the field $\mathbb{R}$. Consider the functions $f_1, f_2, f_3 \in \mathcal{F}$ given by

$$f_1(x) = x^{4/3}, \qquad f_2(x) = e^{2x\ln(9)}, \qquad f_3(x) = 3^{7\pi+4x}$$

Determine whether $\{f_1, f_2, f_3\}$ is linearly dependent or linearly independent, and provide a proof of your answer.

4. (10 points)

(a) Let $A \in M_{m\times n}(\mathbb{R})$. We say that R is a **right inverse** of A if $AR = I_m$. Prove that if A has a right inverse, then $L_A \colon \mathbb{R}^n \to \mathbb{R}^m$ is surjective.

(b) Let V be an inner product space over the real numbers or the complex numbers, where $\langle \cdot, \cdot \rangle$ denotes the inner product. Prove (from the definitions) or disprove (using a counterexample) the following statement:

For any vectors $a, b, c \in V$, if a is orthogonal to b and b is orthogonal to c, then a is orthogonal to c.

5. (12 points) Let $T\colon P_2(\mathbb{R}) \to P_3(\mathbb{R})$ be a linear transformation defined by

$$T(f) = f + f' + f'',$$

where f' denotes the derivative of f. Let $\alpha = \{1, x, x^2\}$ and $\beta = \{1, x, x^2, x^3\}$ denote the standard ordered bases for $P_2(\mathbb{R})$ and $P_3(\mathbb{R})$ respectively.

(a) Find the matrix $[T]_\alpha^\beta$ of T with respect to α and β.

(b) Determine the null space $N(T)$ of T.

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(c) Describe the range $R(T)$ of T by exhibiting a basis for it. You do not need to prove it is a basis.

(d) Is T injective? Justify your answer.

(e) Is T an isomorphism? Justify your answer.

6. (10 points) Let $T\colon M_{2\times 2}(\mathbb{R}) \to M_{2\times 2}(\mathbb{R})$ be a linear operator defined by $T(A) = A^t$.

Please provide full justification for your answers below, even if you have seen this problem before.

(a) Find all the eigenvalues of T.

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(b) Describe the eigenspace of T corresponding to each eigenvalue, by exhibiting a basis for each eigenspace. You do not need to prove they are bases.

(c) Determine whether T is diagonalizable or not. If T is diagonalizable, exhibit a basis for $M_{2\times 2}(\mathbb{R})$ consisting of eigenvectors of T.

7. (10 points) Let T be a linear operator on a vector space V of dimension n.

(a) Prove that, for any integer $k \geq n$, T^k can be expressed as a linear combination of $I, T, T^2, \ldots T^{k-1}$.

Hint: Use the Cayley-Hamilton Theorem. You do not need to use induction, but thinking about the case $k = n$ will give you an idea of what to do in general.

(b) Prove, without using the Cayley-Hamilton Theorem, that the set $\{I, T, T^2, \ldots, T^{n^2-1}, T^{n^2}\}$ is linearly dependent.

Hint: This set is a subset of the vector space $\mathcal{L}(V, V)$ of all linear operators on V.

8. (10 points) Consider the vector space $V = P_2([0,\infty))$ of polynomials of degree at most 2 defined on $[0,\infty)$, with inner product

$$\langle f, g\rangle = \int_0^\infty e^{-x} f(x)g(x)\,dx.$$

Apply the Gram-Schmidt process to the standard basis $\{1, x, x^2\}$ to find an orthogonal basis for V with respect to this inner product.

You may use, without proof, the fact that for any integer $n \geq 0$,

$$\int_0^\infty e^{-x}x^n\,dx = n!$$

9. (10 points) Let V be an inner product space over the real numbers or the complex numbers, where $\langle \cdot, \cdot \rangle$ denotes the inner product. For each $x \in V$, define a linear operator $P_x \colon V \to V$ by $P_x(y) = \langle y, x \rangle x$.

(a) Prove that if $x \in V$ with $\|x\| = 1$, then $(P_x)^2 = P_x$.

(b) Prove or disprove: $\{P_x : x \in V\}$ is a subspace of the vector space $\mathcal{L}(V, V)$ of all linear operators on V.

10. (10 points) Consider the matrix

$$A = \begin{pmatrix} 8 & 10 \\ -5 & -7 \end{pmatrix}.$$

In this problem, you may use, without proof, the fact that A has eigenvalues -2 and 3, which correspond, respectively, to eigenvectors $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ and $\begin{pmatrix} -2 \\ 1 \end{pmatrix}$.

(a) Find matrices D and Q such that $A = QDQ^{-1}$ and D is diagonal.

(b) Find Q^{-1}.

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(c) Find the general solution to the system of differential equations $y'(t) = Dy(t)$. Give explicit expressions for $y_1(t)$ and $y_2(t)$. Your results should involve two arbitrary constants c_1 and c_2.

(d) Find the general solution to the system of differential equations

$$\begin{aligned} x_1'(t) &= 8x_1(t) + 10x_2(t) \\ x_2'(t) &= -5x_1(t) - 7x_2(t) \end{aligned}$$

Extra paper

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