Math 235: Linear Algebra

Midterm Exam 2

November 20, 2014

NAME (please print legibly): Tamar Friedmann
Your University ID Number: Solutions
Please circle your professor's name: **Bobkova** **Friedmann**

- **The presence of calculators, cell phones, iPods and other electronic devices at this exam is strictly forbidden.**
- **Show your work and justify your answers. You may not receive full credit for a correct answer if insufficient work is shown or insufficient justification is given.**
- **Clearly circle or label your final answers, on those questions for which it is appropriate.**

QUESTION	VALUE	SCORE
1	20	
2	10	
3	15	
4	10	
5	15	
6	10	
7	20	
TOTAL	100	

1. (20 points)

Suppose that V is finite dimensional and let $S : V \to V$ and $T : V \to V$ be linear operators. Prove that ST is invertible if and only if both S and T are invertible.

If S and T are invertible then ST is invertible:

$$\begin{array}{ccccc} V & \xrightarrow{T} & V & \xrightarrow{S} & V \\ v & & w & & z \end{array}$$

Since $ST : V \longrightarrow V$, proving ST is 1-1 would imply it's also onto, and proving ST is onto would imply it's also 1-1. So it's enough to prove <u>either</u> 1-1 or onto to prove invertibility. For these solutions to be most useful, I'll do both.

- Show ST is 1-1, ie that if $(ST)(v_1) = (ST)(v_2)$ then $v_1 = v_2$:
  If $(ST)(v_1) = (ST)(v_2)$ then $S(T(v_1)) = S(T(v_2))$, but S is invertible and therefore 1-1, so $T(v_1) = T(v_2)$. But T is also 1-1 so $v_1 = v_2$.

- Show ST in onto; ie for any $z \in V$ $\exists v \in V$ s.t. $(ST)(v) = z$:
  Since S is invertible, it is onto, so $\exists w \in V$ s.t. $S(w) = z$. T is also onto, so $\exists v \in V$ s.t. $T(v) = w$.
  Hence $S(T(v)) = S(w) = z$ so $(ST)(v) = z$.

<u>Another method</u>: pick a basis β of V and write $[S]_\beta$, $[T]_\beta$, $[ST]_\beta$.
Both $[S]_\beta$ and $[T]_\beta$ are invertible matrices, and $[ST]_\beta = [S]_\beta [T]_\beta$
By a theorem, $\operatorname{rank} [S]_\beta [T]_\beta = \operatorname{rank} [T]_\beta$ when $[S]_\beta$ is invertible.
But $\operatorname{rank} [T]_\beta = \dim V$, so $\operatorname{rank} [ST]_\beta = \dim V$ and ST is onto. So as above ST is also 1-1 and so invertible.

If ST is invertible, so are S and T:

<u>S</u>: ST is onto, so $\forall z \in V$, $\exists v \in V$ s.t. $(ST)(v) = z$.
Since $S(w) = z$ for $w = T(v)$, S is onto. Hence it is also 1-1 and so invertible.

<u>T</u>: ST is 1-1 means $\ker(ST) = \{0\}$.
Suppose $v \in \ker T$. Then $T(v) = 0$ so $S(T(v)) = S(0) = 0$.
Hence $v \in \ker ST$ so $v = 0$. Therefore, $\ker T = \{0\}$
so T is 1-1 and hence onto and invertible.

1st Solution

Nov 20 2014 Mid term Exam 2 MTH 235

① ct'd

Another method: use basis β again.

By a theorem, $\operatorname{rank}[S]_\beta[T]_\beta \le \operatorname{rank}[S]_\beta$

and $\operatorname{rank}[S]_\beta[T]_\beta \le \operatorname{rank}[T]_\beta$

ST is invertible so $\operatorname{rank}[S]_\beta[T]_\beta = \dim V$

So $\dim V \le \operatorname{rank}[S]_\beta$

$\dim V \le \operatorname{rank}[T]_\beta$

Since $\operatorname{rank}[S]_\beta$, $\operatorname{rank}[T]_\beta$ cannot be larger than $\dim V$, they must equal $\dim V$.

So both S and T are onto and therefore 1-1 and invertible.

Another method for ST invertible $\Rightarrow S, T$ invertible:

$\operatorname{rk} ST = \underset{\dim V}{n}$ since ST is onto

$\left.\begin{array}{l}\operatorname{rk} ST \le \operatorname{rank} S \\ \operatorname{rank} ST \le \operatorname{rank} T\end{array}\right\}$ by a theorem.

But $\operatorname{rank} S \le n$ so $n = \operatorname{rank} ST \le \operatorname{rank} S \le n \Rightarrow \operatorname{rank} S = n$

same for T.

So S, T are onto. Since onto implies 1-1, S & T are invertible.

Another method for S, T invertible $\Rightarrow ST$ invertible:

S^{-1} and T^{-1} exist. so $(ST)T^{-1}S^{-1} = STT^{-1}S^{-1} = SIS^{-1} = I$

Hence $T^{-1}S^{-1}$ is the inverse of ST, so ST has an inverse.

THIS PAGE
INTENTIONALLY
LEFT BLANK

2. (10 points) In $\mathbb{R}^2$, let $\beta = \{(1,2),(3,4)\}$ and $\beta' = \{(2,4),(4,6)\}$. Find the change of coordinate matrix that changes β' coordinates into β coordinates.

We want the matrix Q that takes (2, 4) to its representation with respect to β, and takes (4, 6) to its representation with respect to β.

$(2,4) = a(1,2) + b(3,4)$

$2 = a + 3b$
$4 = 2a + 4b$

$\left(\begin{array}{cc|c} 1 & 3 & 2 \\ 2 & 4 & 4 \end{array}\right) \to \left(\begin{array}{cc|c} 1 & 3 & 2 \\ 0 & -2 & 0 \end{array}\right)$

$\Rightarrow b = 0, \ a = 2$

$(4,6) = c(1,2) + d(3,4)$

$4 = c + 3d$
$6 = 2c + 4d$

$\left(\begin{array}{cc|c} 1 & 3 & 4 \\ 2 & 4 & 6 \end{array}\right) \to \left(\begin{array}{cc|c} 1 & 3 & 4 \\ 0 & -2 & -2 \end{array}\right)$

$\Rightarrow d = 1, \ c = 1$

So $Q = \begin{pmatrix} 2 & 1 \\ 0 & 1 \end{pmatrix}$

check: $\begin{pmatrix} 2 & 1 \\ 0 & 1 \end{pmatrix} \underbrace{\begin{pmatrix} 1 \\ 0 \end{pmatrix}_{\beta'}}_{\begin{pmatrix} 2 \\ 4 \end{pmatrix}} = \underbrace{\begin{pmatrix} 2 \\ 0 \end{pmatrix}_{\beta}}_{\begin{pmatrix} 2 \\ 4 \end{pmatrix}}$

$\begin{pmatrix} 2 & 1 \\ 0 & 1 \end{pmatrix} \underbrace{\begin{pmatrix} 0 \\ 1 \end{pmatrix}_{\beta'}}_{\begin{pmatrix} 4 \\ 6 \end{pmatrix}} = \underbrace{\begin{pmatrix} 1 \\ 1 \end{pmatrix}_{\beta}}_{\begin{pmatrix} 1 \\ 2 \end{pmatrix} + \begin{pmatrix} 3 \\ 4 \end{pmatrix} = \begin{pmatrix} 4 \\ 6 \end{pmatrix}}$

3. (15 points)

(a) Find all solutions for the system

$$x_1 + 2x_2 + 5x_3 = 1$$
$$x_1 - x_2 - x_3 = 2$$

$$\left(\begin{array}{ccc|c} 1 & 2 & 5 & 1 \\ 1 & -1 & -1 & 2 \end{array}\right) \xrightarrow{r_2 \to r_2 - r_1} \left(\begin{array}{ccc|c} 1 & 2 & 5 & 1 \\ 0 & -3 & -6 & 1 \end{array}\right) \xrightarrow{r_2 \to -r_2/3} \left(\begin{array}{ccc|c} 1 & 2 & 5 & 1 \\ 0 & 1 & 2 & -1/3 \end{array}\right)$$

$$\xrightarrow{r_1 \to r_1 - 2r_2} \left(\begin{array}{ccc|c} 1 & 0 & 1 & 5/3 \\ 0 & 1 & 2 & -1/3 \end{array}\right)$$

$x_3 = t$, a free parameter

$x_2 + 2t = -1/3 \Rightarrow x_2 = -\frac{1}{3} - 2t$

$x_1 + t = 5/3 \Rightarrow x_1 = 5/3 - t$

$$S = \{ (5/3 - t, -1/3 - 2t, t) \mid t \in \mathbb{R} \} = \{ (5/3, -1/3, 0) + t(-1,-2,1) \mid t \in \mathbb{R} \}$$

(b) Write down a product of elementary matrices which transforms the matrix of coefficients from part a) to its reduced row echelon form.

$$\underbrace{\begin{pmatrix} 1 & -2 \\ 0 & 1 \end{pmatrix}}_{r_1 \to r_1 - 2r_2} \underbrace{\begin{pmatrix} 1 & 0 \\ 0 & -1/3 \end{pmatrix}}_{r_2 \to -\frac{1}{3} r_2} \underbrace{\begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}}_{r_2 \to r_2 - r_1}$$

(c) Find all solutions for the system

$$x_1 + 2x_2 + 5x_3 = 0$$
$$x_1 - x_2 - x_3 = 0$$

$$S_{Hom} = \{ t(-1,-2,1) \mid t \in \mathbb{R} \}$$

TF solutions

4. (10 points) Suppose A is a matrix and its row reduced echelon form is

$$\begin{pmatrix} 1 & 2 & 0 & 3 & 8 \\ 0 & 0 & 1 & 2 & 4 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

(a) Compute the dimension of $Null(L_A)$.

rank L_A + nullity $L_A = 5$

rank L_A = number of non-zero rows = 2

$\Rightarrow$ nullity $(L_A) = 5-2 = 3$

(b) If the first column of A is $\begin{pmatrix} 1 \\ 1 \\ 5 \end{pmatrix}$ and the third column of A is $\begin{pmatrix} -1 \\ -1 \\ -1 \end{pmatrix}$, what is A? Show your work.

2^{nd} column $= 2 \cdot 1^{st}$ column $= \begin{pmatrix} 2 \\ 2 \\ 10 \end{pmatrix}$

4^{th} column $= 3 \cdot 1^{st} + 2 \cdot 3^{rd} = 3\begin{pmatrix} 1 \\ 1 \\ 5 \end{pmatrix} + 2\begin{pmatrix} -1 \\ -1 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 13 \end{pmatrix}$

So $A = \begin{pmatrix} 1 & 2 & -1 & 1 & 4 \\ 1 & 2 & -1 & 1 & 4 \\ 5 & 10 & -1 & 13 & 36 \end{pmatrix}$

5^{th} column $= 8 \cdot 1^{st} + 4 \cdot 3^{rd} = 8\begin{pmatrix} 1 \\ 1 \\ 5 \end{pmatrix} + 4\begin{pmatrix} -1 \\ -1 \\ -1 \end{pmatrix} = \begin{pmatrix} 4 \\ 4 \\ 36 \end{pmatrix}$

(c) Find a basis for the vector space spanned by the columns of the matrix A.

$\beta = \left\{ \begin{pmatrix} 1 \\ 1 \\ 5 \end{pmatrix}, \begin{pmatrix} -1 \\ -1 \\ -1 \end{pmatrix} \right\}$

5. (15 points) How many solutions can a homogeneous system of linear equations have? Give an example of a system of two equations in two variables for each case. Explain your examples briefly. You do not have to find the solutions of the systems.

A homogeneous system can have either 1 solution or infinitely many solutions.

1 solution: the matrix has full rank

$$x_1 + x_2 = 0$$
$$x_1 + 2x_2 = 0$$

Only solution is $\{(0,0)\}$

Infinitely many solutions: the matrix does not have full rank

$$x_1 + x_2 = 0$$
$$2x_1 + 2x_2 = 0$$

Solution set is $\{ t(1,-1) \mid t \in \mathbb{R} \}$

TF solutions

6. (10 points) Let

$$A = \begin{pmatrix} 3 & -1 & 2 \\ 2 & 1 & 1 \\ 1 & -3 & 0 \end{pmatrix}.$$

For which $b = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$ does the system $Ax = b$ have a solution?

b must be in $R(L_A)$.

$$\begin{pmatrix} 3 & -1 & 2 \\ 2 & 1 & 1 \\ 1 & -3 & 0 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & -3 & 0 \\ 2 & 1 & 1 \\ 3 & -1 & 2 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & -3 & 0 \\ 0 & 7 & 1 \\ 0 & 8 & 2 \end{pmatrix}$$

rank(A) is clearly 3 so L_A is onto.

Therefore, A is invertible and $A^{-1}b = x$ is a solution for <u>any</u> b.

7. (20 points)

(a) (5 points) Prove that if B is a 2×1 matrix and C is a 1×2 matrix then the 2×2 matrix BC has rank at most one.

By a theorem, $\text{rank}(BC) \le \text{rank} B$, also $\text{rank}(BC) \le \text{rank}\, C$.

The rank of a matrix cannot be larger than its number of rows or columns. Therefore, $\text{rank} B \le 1$; also $\text{rank}\, C \le 1$.

So $\text{rank}(BC) \le 1$.

(b) (15 points) Show that if A is any 2×2 matrix of rank 1 then there exist a 2×1 matrix B and 1×2 matrix C such that $A = BC$. (You will get 40% credit for providing an example.)

If $B = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$ and $C = (c_1 \ c_2)$ then $BC = \begin{pmatrix} b_1c_1 & b_1c_2 \\ b_2c_1 & b_2c_2 \end{pmatrix}$

If A is 2×2 and has rank 1 then its columns (or rows) are multiples of each other.

So A has the form $\begin{pmatrix} a_1 & ka_1 \\ a_2 & ka_2 \end{pmatrix}$ for $a_1, a_2, k \in \mathbb{F}$.

Pick $B = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$, $C = (1, k)$

Then $A = BC$.