Math 235: Linear Algebra

Midterm Exam 1

October 15, 2013

NAME (please print legibly): ______________________________

Your University ID Number: ______________________________

Please circle your professor's name: Friedmann Tucker

- **The presence of calculators, cell phones, iPods and other electronic devices at this exam is strictly forbidden.**
- **Show your work and justify your answers. You may not receive full credit for a correct answer if insufficient work is shown or insufficient justification is given.**
- **Clearly circle or label your final answers, on those questions for which it is appropriate.**

QUESTION	VALUE	SCORE
1	10	
2	25	
3	20	
4	20	
5	15	
6	10	
TOTAL	100	

1. (10 points)

Let $\{v_1, v_2, v_3\}$ be a subset of a vector space V. Suppose that $\{v_1, v_2, v_3\}$ is dependent. Suppose that $v_1 \neq 0$. Prove that we must have $v_2 \in \text{Span}(\{v_1\})$ or $v_3 \in \text{Span}(\{v_1, v_2\})$.

Solution. The set $\{v_1, v_2, v_3\}$ is dependent so there are scalars a, b, and c, not all zero, such that

$$av_1 + bv_2 + cv_3 = 0.$$

Method 1.

If $c \neq 0$ then $v_3 = -\frac{1}{c}(av_1 + bv_2)$ so $v_3 \in \text{Span}(\{v_1, v_2\})$.

If $c = 0$ then $av_1 + bv_2 = 0$ with at least one of a and b nonzero. If $b \neq 0$ then $v_2 = -\frac{a}{b}v_1$ so $v_2 \in \text{Span}(\{v_1\})$ as required. If $b = 0$ then $a \neq 0$ and $av_1 = 0$, contradicting the assumption that $v_1 \neq 0$. So $b \neq 0$.

Method 2.

Suppose $v_2 \notin \text{Span}(\{v_1\})$. Then we show that $c \neq 0$: Suppose $c = 0$, then $av_1 + bv_2 = 0$ so if $b \neq 0$, $v_2 = -\frac{a}{b}v_1$, contradicting that $v_2 \notin \text{Span}(\{v_1\})$. So $b = 0$. But then $av_1 = 0$ with $a \neq 0$, contradicting that $v_1 \neq 0$.

So $c \neq 0$. Then $v_3 \in \text{Span}(\{v_1, v_2\})$ because $v_3 = -\frac{1}{c}(av_1 + bv_2)$.

2. (25 points)

(a) Let V be a vector space of dimension $n \geq 2$. Let W_1 and W_2 be subspaces of V such that $W_1 \neq V$, $W_2 \neq V$, and $W_1 \neq W_2$. Show that $\dim(W_1 \cap W_2) \leq \dim V - 2$.

Solution. Since $W_1, W_2 \neq V$, we see that $\dim W_1 < V$ and $\dim W_2 < V$, as proved in class and in the book. Thus (since dimensions are integers), we have

$$\dim W_1 \leq \dim V - 1 \quad \text{and} \quad \dim W_2 \leq \dim V - 1 \tag{1}$$

Since $W_1 \neq W_2$, we see that either there is a vector in W_1 that is not in W_2 or a vector in W_2 that is not in W_1. Thus, we have $W_1 \cap W_2 \neq W_1$ or $W_1 \cap W_2 \neq W_2$. This means that $\dim(W_1 \cap W_2) < \dim W_1$ or $\dim(W_1 \cap W_2) < \dim W_2$. By (1), we then have $\dim(W_1 \cap W_2) \leq \dim V - 2$, as desired.

(b) Let V the space of all functions $f : \mathbb{R} \longrightarrow \mathbb{R}$, and let W be the set of all functions f such that $f(1) = -f(2)$. Show that W is a subspace of V.

Solution. We will check that 0 is in W, that W is closed under addition, and that W is closed under scalar multiplication.

The zero element of V is the function 0_V such that $0_V(x) = 0$ for all $x \in \mathbb{R}$. Clearly $0_V(1) = 0 = -0 = -0_V(2)$, so $0_V \in W$.

Now, suppose $f, g \in W$. Then

$$(f+g)(1) = f(1) + g(1) = -f(2) - g(2) = -(f+g)(2)$$

so $f + g \in W$. Thus, W is closed under addition.

Now, let $a \in \mathbb{R}$ and $f \in W$. Then

$$(af)(1) = af(1) = a(-f(2)) = -af(2) = -(af)(2).$$

Thus, W is closed under scalar multiplication.

We see then that W is a subspace of V.

(c) Let $T : \mathbb{R}^2 \longrightarrow \mathbb{R}^3$ be a linear transformation. Let W be the set of all $v \in \mathbb{R}^2$ such that $T(v) = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$. Is W a subspace of $\mathbb{R}^2$? Explain your answer carefully.

Solution. No, it is not. We do not have $0 \in W$ since $T(0) = 0$, which is not $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$.

In fact, W is also not closed under scalar multiplication or addition either, as you may easily check.

3. (20 points)

Let $T : P_1(\mathbb{R}) \longrightarrow P_1(\mathbb{R})$ (here $P_1(\mathbb{R})$ is the set of polynomials of degree at most 1 with coefficients in $\mathbb{R}$ as usual) be the linear map such that $T(x+1) = x$ and $T(x-1) = 5x$.

(a) Find $T(1)$.

Solution. Since

$$1 = \frac{(x+1)-(x-1)}{2},$$

we have

$$T(1) = T\left(\frac{(x+1)-(x-1)}{2}\right) = \frac{1}{2}T(x+1) - \frac{1}{2}T(x-1) = \frac{x}{2} - \frac{5x}{2} = -2x.$$

(b) Is T one-one? Explain your answer.

Solution. No: $T(5(x+1) = 5T(x+1) = 5x = T(x-1)$ but $5(x+1) \neq (x-1)$.

(c) Calculate $\dim \mathrm{R}(T)$.

Solution.

The set $\{x+1, x-1\}$ forms a basis for $P_1(\mathbb{R})$. So

$$\mathrm{R}(T) = \mathrm{Span}\{T(x+1), T(x-1)\} = \mathrm{Span}\{x, 5x\} = \mathrm{Span}\{x\},$$

hence $\dim \mathrm{R}(T) = 1$.

(d) Let β be the ordered basis $\{1, x\}$ for $P_1(\mathbb{R})$. Write down the matrix $[T]_\beta^\beta$.

Solution. $T(1) = -2x$ from (a).

$$T(x) = T\left(\frac{(x+1)+(x-1)}{2}\right) = \frac{1}{2}T(x+1) + \frac{1}{2}T(x-1) = \frac{x}{2} + \frac{5x}{2} = 3x$$

So

$$[T]_\beta^\beta = \begin{pmatrix} 0 & 0 \\ -2 & 3 \end{pmatrix}$$

4. (20 points)

Let V be a vector space and let $T : V \longrightarrow V$ be a linear transformation.

(a) Suppose that $\{v_1, v_2\}$ are dependent. Show that $\{T(v_1), T(v_2)\}$ must also be dependent.

Solution.

Since $\{v_1, v_2\}$ are dependent, there are a, b, not both zero, such that $av_1 + bv_2 = 0$. For any linear transformation, $T(0) = 0$, hence $T(av_1 + bv_2) = 0$. Since T is linear, $aT(v_1) + bT(v_2) = 0$ with a, b not both zero, so $\{T(v_1), T(v_2)\}$ is dependent.

(b) True or false and explain: Suppose that $\{v_1, v_2\}$ are independent. Then $\{T(v_1), T(v_2)\}$ must also be independent.

Solution. False. Counterexample: Let T be the zero transformation. Then $T(v_1) = T(v_2) = 0$ for any v_1 and v_2.

(c) Suppose now that $\dim V = 3$. Show that we must have $\mathrm{N}(T) \neq \mathrm{R}(T)$.

Solution.

By the rank-nullity theorem, $\dim \mathrm{N}(T) + \dim \mathrm{R}(T) = 3$. If $\mathrm{N}(T) = \mathrm{R}(T)$ then $\dim \mathrm{N}(T) = \dim \mathrm{R}(T) = 3/2$, which is impossible since dimension is an integer. So $\mathrm{N}(T) \neq \mathrm{R}(T)$.

(d) Suppose again that $\dim V = 3$. True or false and explain: if $T \neq 0$, then $T^2 \neq 0$.

Solution. False. Counterexample: Let $\{e_1, e_2, e_3\}$ be a basis for V. Let $T(e_1) = e_2, T(e_2) = 0, T(e_3) = 0$. So $T \neq 0$. But $T^2(e_1) = T(T(e_1)) = T(e_2) = 0$, and more trivially, $T^2(e_2) = T^2(e_3) = 0$. So $T^2 = 0$.

5. (15 points)

(a) Let $S = \left\{ \begin{pmatrix} 1 \\ 3 \end{pmatrix}, \begin{pmatrix} 0 \\ 3 \end{pmatrix} \right\}$. Is S linearly independent? Does S span $\mathbb{R}^2$? Is S a basis for $\mathbb{R}^2$? (Explain your answers.)

Solution. Neither of these vectors is a multiple of the other and there are two of them. Thus, they are linearly independent, span $\mathbb{R}^2$, and are a basis for $\mathbb{R}^2$.

(b) Let $S = \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 5 \\ 4 \end{pmatrix} \right\}$. Is S linearly independent? Does S span $\mathbb{R}^2$? Is S a basis for $\mathbb{R}^2$? (Explain your answers.)

Solution. There are three vectors here and the dimension of $\mathbb{R}^2$ is 2, so they are clearly *not* linearly independent. Since no vector is a multiple of another vector, the dimension of the span is 2, so they do span all of $\mathbb{R}^2$. They are not a basis, since they are not linearly independent.

(c) Let $S = \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\}$. Is S linearly independent? Does S span $\mathbb{R}^2$? Is S a basis for $\mathbb{R}^2$? (Explain your answers.)

Solution. The set S consists of a single nonzero vector, so it is clearly linearly independent. It cannot span $\mathbb{R}^2$ since $\mathbb{R}^2$ has dimension 2. It is not a basis for $\mathbb{R}^2$ since it does not span $\mathbb{R}^2$.

6. (10 points) Suppose that $\{u, v\}$ is a basis for a vector space V. Show that $\{u+v, u+2v\}$ is also a basis for V.

Solution. Since V has dimension 2, any set of two elements that is linearly independent will be a basis for V, as we've proved in class and in the book. Thus, it will suffice to show that $\{u + v, u + 2v\}$ is linearly independent. (Note: It would also suffice to show that $\{u + v, u + 2v\}$ spans V since any set of two vectors that spans V will be a basis.)

Suppose that we have $a(u+v)+b(u+2v) = 0$ for scalars a and b. Then $(a+b)u+(a+2b)v = 0$. Since $\{u, v\}$ is linearly independent, this means

$$\begin{aligned} a + b &= 0 \\ a + 2b &= 0 \end{aligned} \tag{2}$$

Subtracting the first equation from the second we obtain $b = 0$. Substituting then gives $a = 0$.

Thus, we see that the only scalars a, b such that $a(u + v) + b(u + 2v) = 0$ are $a = b = 0$. Therefore, $\{u + v, u + 2v\}$ is linearly independent and our proof is done.