Math 235: Abstract Algebra

Midterm Exam 1

October 15, 2013

NAME (please print legibly): ______________________________

Your University ID Number: ______________________________

Please circle your professor's name: **Friedmann** **Tucker**

- **The presence of calculators, cell phones, iPods and other electronic devices at this exam is strictly forbidden.**
- **Show your work and justify your answers. You may not receive full credit for a correct answer if insufficient work is shown or insufficient justification is given.**
- **Clearly circle or label your final answers, on those questions for which it is appropriate.**

QUESTION	VALUE	SCORE
1	10	
2	25	
3	20	
4	20	
5	15	
6	10	
TOTAL	100	

1. (10 points)

Let $\{v_1, v_2, v_3\}$ be a subset of a vector space V. Suppose that $\{v_1, v_2, v_3\}$ is dependent. Suppose that $v_1 \neq 0$. Prove that we must have $v_2 \in \text{Span}(\{v_1\})$ or $v_3 \in \text{Span}(\{v_1, v_2\})$.

2. (25 points)

(a) Let V be a vector space of dimension $n \geq 2$. Let W_1 and W_2 be subspaces of V such that $W_1 \neq V$, $W_2 \neq V$, and $W_1 \neq W_2$. Show that $\dim(W_1 \cap W_2) \leq \dim V - 2$.

(b) Let V the space of all functions $f : \mathbb{R} \longrightarrow \mathbb{R}$, and let W be the set of all functions f such that $f(1) = -f(2)$. Show that W is a subspace of V.

(c) Let $T : \mathbb{R}^2 \longrightarrow \mathbb{R}^3$ be a linear transformation. Let W be the set of all $v \in \mathbb{R}^2$ such that $T(v) = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$. Is W a subspace of $\mathbb{R}^2$? Explain your answer carefully.

3. (20 points)

Let $T : P_1(\mathbb{R}) \longrightarrow P_1(\mathbb{R})$ (here $P_1(\mathbb{R})$ is the set of polynomials of degree at most 1 with coefficients in $\mathbb{R}$ as usual) be the linear map such that $T(x+1) = x$ and $T(x-1) = 5x$.

(a) Find $T(1)$.

(b) Is T one-one? Explain your answer.

(c) Calculate $\dim \mathrm{R}(T)$.

(d) Let β be the ordered basis $\{1, x\}$ for $P_1(\mathbb{R})$. Write down the matrix $[T]_\beta^\beta$.

4. (20 points)

Let V be a vector space and let $T : V \longrightarrow V$ be a linear transformation.

(a) Suppose that $\{v_1, v_2\}$ are dependent. Show that $\{T(v_1), T(v_2)\}$ must also be dependent.

(b) True or false and explain: Suppose that $\{v_1, v_2\}$ are independent. Then $\{T(v_1), T(v_2)\}$ must also be independent.

(c) Suppose now that $\dim V = 3$. Show that we must have $\mathrm{N}(T) \neq \mathrm{R}(T)$.

(d) Suppose again that $\dim V = 3$. True or false and explain: if $T \neq 0$, then $T^2 \neq 0$.

5. (15 points)

(a) Let $S = \left\{ \begin{pmatrix} 1 \\ 3 \end{pmatrix}, \begin{pmatrix} 0 \\ 3 \end{pmatrix} \right\}$. Is S linearly independent? Does S span $\mathbb{R}^2$? Is S a basis for $\mathbb{R}^2$? (Explain your answers.)

(b) Let $S = \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 5 \\ 4 \end{pmatrix} \right\}$. Is S linearly independent? Does S span $\mathbb{R}^2$? Is S a basis for $\mathbb{R}^2$? (Explain your answers.)

(c) Let $S = \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\}$. Is S linearly independent? Does S span $\mathbb{R}^2$? Is S a basis for $\mathbb{R}^2$? (Explain your answers.)

6. (10 points) Suppose that $\{u, v\}$ is a basis for a vector space V. Show that $\{u+v, u+2v\}$ is also a basis for V.