
These solutions are not intended to be exhaustive, but they should be sufficient for anyone who has already tried to work through the problems. Some problems may have multiple different solutions, so yours may still be correct even if it is different from the solution appearing here.

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1. For each part, simply try to verify the two parts of the definition of a linear transformation.

   (a) This map $\boxed{\text{is a linear transformation}}$, and in fact is left-multiplication by $\begin{bmatrix} 1 & -1 \\ 2 & 0 \\ 1 & 1 \end{bmatrix}$.

   - [T1]: We have $T(x_1 + x_2, y_1 + y_2) = \langle x_1 + x_2 - y_1 - y_2, 2x_1 + 2x_2, x_1 + x_2 + y_1 + y_2 \rangle = T(x_1, y_1) + T(x_2, y_2)$.
   - [T2]: We have $T(cx, cy) = \langle cx - cy, 2cx, cx + cy \rangle = cT(x, y)$.

   (b) This map $\boxed{\text{is not a linear transformation}}$, as it fails both [T1] and [T2]. For example, if $A = I_2$ then $T(A) = I_2$, so $T(2A) = 4I_2$ is not equal to $2T(A)$.

   (c) This map $\boxed{\text{is a linear transformation}}$.

   - [T1]: We have $T(p_1 + p_2) = (p_1 + p_2)'(x + 1) = p_1'(x + 1) + p_2'(x + 1) = T(p_1) + T(p_2)$.
   - [T2]: We have $T(cp) = (cp)'(x + 1) = c\, p'(x + 1) = cT(p)$.

2.

   (a) $\boxed{\text{True}}$: For example, $T(2x + x^2 + x^3) = \langle 2, 2, 6 \rangle$. (In fact, T is onto.)

   (b) $\boxed{\text{False}}$: The set $\{T(\mathbf{v}_1), \dots, T(\mathbf{v}_n)\}$ is only a spanning set for $\text{im}(T)$, and may be linearly dependent. (For example, T could be the zero transformation.)

   (c) $\boxed{\text{True}}$: Since $T(A) = 0$ only when $A = 0$, and $T(\frac{1}{2}A^T) = A$, T is one-to-one and onto. Alternatively, T has an inverse function $T^{-1}(A) = \frac{1}{2}A^T$, so it is an isomorphism.

   (d) $\boxed{\text{False}}$: Although the given dimensions are consistent with the nullity-rank theorem, the image of T must be a subspace of $\mathbb{R}^2$ and therefore cannot have dimension 3.

   (e) $\boxed{\text{False}}$: The derivative map $D : P(\mathbb{R}) \to P(\mathbb{R})$ is an example of an onto map that is not one-to-one. (If V were finite-dimensional, the statement would be true.)

   (f) $\boxed{\text{True}}$: This is a theorem. To summarize, $\langle T^*(\mathbf{v}), \mathbf{w} \rangle = \overline{\langle \mathbf{w}, T^*(\mathbf{v}) \rangle} = \overline{\langle T(\mathbf{w}), \mathbf{v} \rangle} = \langle \mathbf{v}, T(\mathbf{w}) \rangle$, so T satisfies the requirements of $(T^*)^*$. Since the adjoint is unique, we conclude $(T^*)^* = T$.

   (g) $\boxed{\text{True}}$: If $\dim(V) = n$ and $\dim(W) = m$, then $\mathcal{L}(V, W)$ and $\mathcal{L}(W, V)$ both have dimension mn, so they are isomorphic. (In fact they are still isomorphic in the infinite-dimensional case.)

   (h) $\boxed{\text{False}}$: In general, $[I]_\alpha^\beta$ is the change-of-basis matrix from α-coordinates to β-coordinates, so the given statement is only true when $\alpha = \beta$.

   (i) $\boxed{\text{True}}$: The cancellation was performed in the correct order.

   (j) $\boxed{\text{True}}$: This is an example of change of basis: the desired Q is $[I]_\beta^\alpha$, since then $Q^{-1} = [I]_\alpha^\beta$, and $[T]_\beta^\beta = [I]_\alpha^\beta [T]_\alpha^\alpha [I]_\beta^\alpha = Q^{-1}[T]_\alpha^\alpha Q$.

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3. If $p(x) = a + bx + cx^2$, observe that $T(p) = \langle a + b + c, b + 2c, a + 2b + 3c \rangle$.

   (a) Notice that $T(p) = \langle 0, 0, 0 \rangle$ precisely when $p(1) = p'(1) = p(1) + p'(1) = 0$, so $p(1) = p'(1) = 0$, or equivalently, $a + b + c = b + 2c = 0$. Solving yields $p = a(x - 1)^2$, so a basis is given by $\boxed{\{(x - 1)^2\}}$.

(b) Since $\beta = \{1, x, x^2\}$ is a basis of $P_2(\mathbb{R})$, $\{T(1), T(x), T(x^2)\}$ will span the image of T. We compute $T(1) = \langle 1, 0, 1\rangle$, $T(x) = \langle 1, 1, 2\rangle$, and $T(x^2) = \langle 2, 1, 3\rangle$. Since the first two vectors are linearly independent but the third is a linear combination (the sum), a basis is given by $\boxed{\{\langle 1, 0, 1\rangle, \langle 1, 1, 2\rangle\}}$.

(c) The columns are the coefficients of $T(1) = \langle 1, 0, 1\rangle$, $T(x) = \langle 1, 1, 2\rangle$, $T(x^2) = \langle 2, 1, 3\rangle$ with respect to the basis γ, so the matrix is $[T]_\beta^\gamma = \boxed{\begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 1 \\ 1 & 2 & 3 \end{bmatrix}}$.

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4. If $T(\mathbf{v}) = \mathbf{0}$ then $\mathbf{v} = T^3(\mathbf{v}) = T^2(\mathbf{0}) = \mathbf{0}$, so T is one-to-one. Furthermore, $T(T^2\mathbf{v}) = \mathbf{v}$, so T is onto. Hence T is an isomorphism.
Alternatively, if $T^3 = I$, then $T(T^2) = I = (T^2)T$, so T^2 is a two-sided inverse of T. Therefore, T is an isomorphism and hence both one-to-one and onto.

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5. Suppose $T(\mathbf{v}) = \mathbf{0}$. Writing $\mathbf{v} = a_1\mathbf{v}_1 + \cdots + a_n\mathbf{v}_n$ we see $T(\mathbf{v}) = a_1T(\mathbf{v}_1) + \cdots + a_nT(\mathbf{v}_n)$, so since γ is linearly independent, all the coefficients are zero, so $\mathbf{v} = \mathbf{0}$ and T is one-to-one. Furthermore, T is onto, since if $\mathbf{w} = a_1T(\mathbf{v}_1) + \cdots + a_nT(\mathbf{v}_n)$ then $\mathbf{w} = T(a_1\mathbf{v}_1 + \cdots + a_n\mathbf{v}_n)$. Thus, T is one-to-one and onto, hence an isomorphism.
Alternatively, since γ is a basis of W, there exists a linear transformation $R : W \to V$ with $R(T(\mathbf{v}_i)) = \mathbf{v}_i$ for each $1 \le i \le n$, since a linear transformation can be specified arbitrarily on a basis. Then R is an inverse transformation for T, so T is an isomorphism.
Alternatively, the given information implies that $[T]_\beta^\gamma = I_n$. Since the associated matrix is invertible, T is invertible, and therefore an isomorphism.

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6. Note that this problem is partly a special case of problem II.C on homework 8.

(a) Suppose $\mathbf{w}$ is in $\text{im}(T)$. Then there exists $\mathbf{v}$ with $\mathbf{w} = T(\mathbf{v})$. Then $T(\mathbf{w}) = T(T(\mathbf{v})) = \mathbf{0}$, meaning that $\mathbf{w}$ is in $\ker(T)$. Thus, $\text{im}(T)$ is contained in $\ker(T)$.

(b) By part (a), $\dim(\text{im}T) \le \dim(\ker T)$, and by nullity-rank, $\dim(\text{im}T) + \dim(\ker T) = 2$. Thus, $\dim(\text{im}T)$ is either 0 or 1. But the dimension of the image cannot be zero, since this would imply that T is the zero transformation. Thus, $\dim(\text{im}T) = 1$.

(c) Since $\{\mathbf{v}, \mathbf{w}\}$ has size $2 = \dim(\mathbb{R}^2)$ it is enough to show that $\mathbf{v}$ and $\mathbf{w}$ are linearly independent. But if $\mathbf{0} = a\mathbf{v} + b\mathbf{w}$ then $\mathbf{0} = T(\mathbf{0}) = T(a\mathbf{v} + b\mathbf{w}) = aT(\mathbf{v}) + bT(\mathbf{w}) = b\mathbf{v}$, so since $\mathbf{v}$ is nonzero, $b = 0$. Then $a\mathbf{v} = \mathbf{0}$ so $a = 0$. So $\{\mathbf{v}, \mathbf{w}\}$ is linearly independent, hence a basis.

(d) Since $T(\mathbf{v}) = \mathbf{0} = 0\mathbf{v} + 0\mathbf{w}$ and $T(\mathbf{w}) = \mathbf{v} = 1\mathbf{v} + 0\mathbf{w}$, the matrix is $[T]_\beta^\beta = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ as claimed.

7.

(a) Suppose $\mathbf{v}$ is in $\text{im}(T)$ and $\mathbf{w}$ is in $\ker(T^*)$: then there exists some $\mathbf{y}$ with $T(\mathbf{y}) = \mathbf{v}$ and $T^*(\mathbf{w}) = \mathbf{0}$. Then $\langle \mathbf{v}, \mathbf{w}\rangle = \langle T(\mathbf{y}), \mathbf{w}\rangle = \langle \mathbf{y}, T^*(\mathbf{w})\rangle = \langle \mathbf{y}, \mathbf{0}\rangle = 0$, so $\mathbf{v}$ and $\mathbf{w}$ are orthogonal.

(b) Since β is an orthonormal basis, the matrix associated to T^* is the conjugate-transpose of the matrix associated to T, so, explicitly, $[T^*]_\beta^\beta = \boxed{\begin{bmatrix} 1 & 4 \\ -3i & 2+i \end{bmatrix}}$.
