MTH 235: Linear Algebra

Midterm 2

April 4, 2017

NAME (please print legibly): ______________________________

Your University ID Number: ______________________________

Circle your instructor:

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- **The presence of electronic devices (including calculators), books, or formula cards/sheets at this exam is strictly forbidden.**
- **Show all work and justify all answers. You MUST provide complete, clear answers for each problem, and justify each step in your proofs.**
- **You may appeal to any theorems, propositions, etc. covered at any point in the course, but please make clear what results you are using.**
- **In problems with multiple parts, you may use the results of previous parts in later parts, EVEN if you did not solve the earlier parts correctly.**
- **You are responsible for checking that this exam has all 7 pages.**

Pledge of Honesty

I affirm that I will not give or receive any unauthorized help on this exam, and that all work will be my own.

Signature: ______________________________

QUESTION	VALUE	SCORE
1	15	
2	25	
3	14	
4	8	
5	10	
6	16	
7	12	
TOTAL	100	

1. (15 points) For each map $T : V \to W$, determine whether or not T is a linear transformation. If so, prove it, and if not, identify (at least) one property that fails.

(a) $T : \mathbb{R}^2 \to \mathbb{R}^3$, where $T(x, y) = \langle x - y, 2x, x + y \rangle$.

(b) $T : M_{3\times3}(\mathbb{R}) \to M_{3\times3}(\mathbb{R})$, where $T(A) = A^T A$.

(c) $T : P(\mathbb{C}) \to P(\mathbb{C})$, where $T(p(x)) = p'(x + 1)$.

2. (25 points) For each of the following, circle the correct response (there is no partial credit or penalty for wrong answers, and no work is required). Assume $T : V \to W$ is a linear transformation, where V and W are **not necessarily finite-dimensional**.

True **False** The vector $\mathbf{v} = \langle 2, 2, 6\rangle$ is in the image of the map $T : P_4(\mathbb{R}) \to \mathbb{R}^3$ with $T(p) = \langle p'(0), p''(0), p'''(0)\rangle$.

True **False** If $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ is a basis of V, then $\{T(\mathbf{v}_1), \ldots, T(\mathbf{v}_n)\}$ is a basis for $\mathrm{im}(T)$.

True **False** The linear transformation $T : M_{2\times3}(\mathbb{R}) \to M_{3\times2}(\mathbb{R})$ given by $T(A) = 2A^T$ is one-to-one and onto.

True **False** There exists a linear map $T : \mathbb{R}^5 \to \mathbb{R}^2$ with nullity 2 and rank 3.

True **False** If $T : V \to V$ is linear, then T is one-to-one if and only if T is onto.

True **False** If T^* exists, then $(T^*)^*$ necessarily exists and equals T.

Now assume that the vector spaces V and W are finite-dimensional, that α, β, and γ are ordered bases of V, V, and W respectively, and that S and T are linear transformations.

True **False** $\mathcal{L}(V, W)$ is isomorphic to $\mathcal{L}(W, V)$.

True **False** If $I : V \to V$ is the identity map, then $[I]_\alpha^\beta$ is always the identity matrix.

True **False** If $S : V \to W$ and $T : W \to V$, then $[ST]_\gamma^\gamma = [S]_\beta^\gamma [T]_\gamma^\beta$.

True **False** For any $T : V \to V$, there always exists an invertible matrix Q such that $[T]_\beta^\beta = Q^{-1}[T]_\alpha^\alpha Q$.

3. (14 points) Let $T : P_2(\mathbb{R}) \to \mathbb{R}^3$ be the linear transformation with

$$T(p) = \langle p(1),\, p'(1),\, p(1) + p'(1) \rangle .$$

(a) Find a basis for the kernel of T.

(b) Find a basis for the image of T.

(c) With the ordered bases $\beta = \{1, x, x^2\}$ and $\gamma = \{\langle 1, 0, 0 \rangle, \langle 0, 1, 0 \rangle, \langle 0, 0, 1 \rangle\}$ find the associated matrix $[T]_\beta^\gamma$.

4. (8 points) Suppose $T : V \to V$ is a linear transformation with the property that T^3 is the identity transformation. Prove that T is one-to-one and onto.

5. (10 points) Suppose that V is finite-dimensional with ordered basis $\beta = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$, and that $T : V \to W$ is linear. If $\gamma = \{T(\mathbf{v}_1), \ldots, T(\mathbf{v}_n)\}$ is a basis of W, prove that T is an isomorphism.

6. (16 points) Suppose $T : \mathbb{R}^2 \to \mathbb{R}^2$ is a linear transformation such that T^2 is the zero transformation, but T is not.

(a) Show that the kernel of T contains the image of T.

(b) Show that $\dim(\text{im}(T)) = 1$.

(c) Let $\mathbf{v}$ be a nonzero vector in $\text{im}(T)$, where $T(\mathbf{w}) = \mathbf{v}$. Prove that $\beta = \{\mathbf{v}, \mathbf{w}\}$ is a basis of $\mathbb{R}^2$. (Hint: Apply T to a linear dependence.)

(d) With $\beta = \{\mathbf{v}, \mathbf{w}\}$ as in part (c), show that the matrix $[T]_\beta^\beta = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$.

7. (12 points) Let V be an inner product space (not necessarily finite-dimensional) and $T : V \to V$ be a linear transformation possessing an adjoint T^*.

(a) If $\mathbf{v}$ is any vector in $\mathrm{im}(T)$ and $\mathbf{w}$ is any vector in $\ker(T^*)$, prove that $\mathbf{v}$ and $\mathbf{w}$ are orthogonal.

(b) Let β be the standard orthonormal basis of $\mathbb{C}^3$. Suppose that $T : \mathbb{C}^3 \to \mathbb{C}^3$ is the linear transformation with associated matrix $[T]_\beta^\beta = \begin{bmatrix} 1 & 3i \\ 4 & 2-i \end{bmatrix}$. What is the matrix $[T^*]_\beta^\beta$ associated to the adjoint T^*?