Math 235, Spring 2017 (Dummit/M$^{\text{c}}$Tague) ~ Linear Algebra ~ Midterm 1 Solutions

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These solutions are not intended to be exhaustive, but they should be sufficient for anyone who has already tried to work through the problems. Some problems may have multiple different solutions, so yours may still be correct even if it is different from the solution appearing here.

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1. We prove this by induction on n. For the base case we take $n = 2$: then indeed $1 \cdot 2 = \dfrac{2^3 - 2}{3}$. For the inductive step, suppose that $1 \cdot 2 + 2 \cdot 3 + \cdots + (n-1) \cdot n = \dfrac{n^3 - n}{3}$. Then

$$\begin{aligned} 1 \cdot 2 + 2 \cdot 3 + \cdots + (n-1) \cdot n + n \cdot (n+1) &= [1 \cdot 2 + 2 \cdot 3 + \cdots + (n-1) \cdot n] + n \cdot (n+1) \\ &= \frac{n^3 - n}{3} + n(n+1) \\ &= \frac{n^3 - n + 3n^2 + 3n}{3} \\ &= \frac{n^3 + 3n^2 + 2n}{3} = \frac{(n+1)^3 - (n+1)}{3} \end{aligned}$$

as required.

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2. If $z = ri$, then $\overline{z} = -ri = -z$. Conversely, if $z = a + bi$, we see $\overline{z} = a - bi$ while $-z = -a - bi$, and these are equal precisely when $a = 0$.

3.

(a) True: the set of vectors in both W_1 and W_2 is the same as the intersection $W_1 \cap W_2$, which we proved was a subspace of V.

(b) True: it satisfies the three components of the subspace criterion.

(c) False: since $P_4(\mathbb{R})$ has dimension 4, any spanning set must contain at least 4 vectors; the given set has only 3.

(d) False: the zero space has dimension 0. (The dimension of any other space is positive.)

(e) False: while it is true that any spanning set must contain 3 vectors, there are certainly sets with many vectors that do not span V. An example is $V = \mathbb{R}^2$ with the set $\{\langle 1, 0 \rangle, \langle 2, 0 \rangle, \langle 3, 0 \rangle, \langle 4, 0 \rangle\}$.

(f) True: any basis must have exactly 3 elements.

(g) False: since $\mathbb{R}^3$ has dimension 3, any set of more than 3 vectors is linearly dependent.

(h) True: we have $\langle 3\mathbf{x}, 2\mathbf{x} \rangle = 6 \langle \mathbf{x}, \mathbf{x} \rangle$, and $\langle \mathbf{x}, \mathbf{x} \rangle \geq 0$ by the positive-definiteness property.

(i) True: this is the triangle inequality.

(j) True: these vectors are orthogonal, each of them has length 1, and there are three of them (meaning that they form a linearly independent set in the 3-dimensional vector space $\mathbb{R}^3$, so they are a basis).

4.

(a) We just check the parts of the subspace criterion. Note that the vectors in S have the form $\langle x_1, x_2, x_3, x_3, x_1 + x_2 \rangle$.

- [S1]: The zero vector satisfies both conditions.
- [S2]: If $\mathbf{v} = \langle x_1, x_2, x_3, x_3, x_1 + x_2 \rangle$ and $\mathbf{w} = \langle y_1, y_2, y_3, y_3, y_1 + y_2 \rangle$ then $\mathbf{v} + \mathbf{w} = \langle x_1 + y_1, x_1 + y_2, x_3 + y_3, x_3 + y_3, (x_1 + y_1) + (x_2 + y_2) \rangle$ which is of the desired form.

- [S3]: If $\mathbf{v} = \langle x_1, x_2, x_3, x_3, x_1 + x_2\rangle$ then $c\mathbf{v} = \langle cx_1, cx_2, cx_3, cx_3, cx_1 + cx_2\rangle$ which is of the desired form.

(b) As noted above, the vectors in S are those of the form $\langle x_1, x_2, x_3, x_3, x_1 + x_2\rangle$. Since $\langle x_1, x_2, x_3, x_3, x_1 + x_2\rangle = x_1 \langle 1,0,0,0,1\rangle + x_2 \langle 0,1,0,0,1\rangle + x_3 \langle 0,0,1,1,0\rangle$, we see that $\langle 1,0,0,0,1\rangle, \langle 0,1,0,0,1\rangle, \langle 0,0,1,1,0\rangle$ span S. Furthermore, since they are clearly linearly independent, they are a basis for S. So we get the basis $\boxed{\{\langle 1,0,0,0,1\rangle, \langle 0,1,0,0,1\rangle, \langle 0,0,1,1,0\rangle\}}$ and get $\dim(S) = \boxed{3}$.

5.

(a) Suppose $\mathbf{w}$ is in V. If S spans V, then there exist scalars $a_1, a_2, \ldots, a_n$ such that $\mathbf{w} = a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + \cdots + a_n\mathbf{v}_n$. In order to show that T spans V, we need to show that there exist scalars $b_1, b_2, \ldots, b_n$ such that $\mathbf{w} = b_1(\mathbf{v}_1 - \mathbf{v}_2) + b_2(\mathbf{v}_2 - \mathbf{v}_3) + \cdots + b_{n-1}(\mathbf{v}_{n-1} - \mathbf{v}_n) + b_n\mathbf{v}_n$.
Expanding and collecting terms yields $\mathbf{w} = b_1\mathbf{v}_1 + (b_2 - b_1)\mathbf{v}_2 + (b_3 - b_2)\mathbf{v}_3 + \cdots + (b_n - b_{n-1})\mathbf{v}_n$.
Comparing this to the linear combination we had for $\mathbf{w}$ above, we should try $b_1 = a_1$, $b_2 - b_1 = a_2$, $b_3 - b_2 = a_3$, ... , $b_n - b_{n-1} = a_n$. This yields $b_1 = a_1$, $b_2 = a_1 + a_2$, $b_3 = a_1 + a_2 + a_3$, ... , $b_n = a_1 + a_2 + \cdots + a_n$.
So, by the calculation above, we can write $\mathbf{w} = a_1(\mathbf{v}_1 - \mathbf{v}_2) + (a_1 + a_2)(\mathbf{v}_2 - \mathbf{v}_3) + \cdots + (a_1 + \cdots + a_n)\mathbf{v}_n$, meaning that $\mathbf{w}$ is in span(T).

Remark In part (a), many students showed the reverse implication (if T spans V then S spans V) by starting with $\mathbf{w}$ as a linear combination of elements in T, expanding as

$$\begin{aligned}\mathbf{w} &= b_1(\mathbf{v}_1 - \mathbf{v}_2) + b_2(\mathbf{v}_2 - \mathbf{v}_3) + \cdots + b_{n-1}(\mathbf{v}_{n-1} - \mathbf{v}_n) + b_n\mathbf{v}_n \\ &= b_1\mathbf{v}_1 + (b_2 - b_1)\mathbf{v}_2 + (b_3 - b_2)\mathbf{v}_3 + \cdots + (b_n - b_{n-1})\mathbf{v}_n\end{aligned}$$

and then concluding that the vector $\mathbf{w}$ was in span(S).

(b) Suppose that we had a dependence $b_1(\mathbf{v}_1 - \mathbf{v}_2) + b_2(\mathbf{v}_2 - \mathbf{v}_3) + \cdots + b_{n-1}(\mathbf{v}_{n-1} - \mathbf{v}_n) + b_n\mathbf{v}_n = \mathbf{0}$. Expanding like in part (a), we see that $b_1\mathbf{v}_1 + (b_2 - b_1)\mathbf{v}_2 + (b_3 - b_2)\mathbf{v}_3 + \cdots + (b_n - b_{n-1})\mathbf{v}_n = \mathbf{0}$. But now since S is linearly independent, each coefficient must be zero: this gives $b_1 = b_2 - b_1 = b_3 - b_3 = \cdots = b_n - b_{n-1} = 0$, so clearly each of $b_1, b_2, \ldots, b_n$ must be zero.

(c) If S is a basis for V, then since S spans V, part (a) implies T spans V. Also, since S is linearly independent, part (b) implies T is linearly independent. Then T spans V and is linearly independent, so it is a basis.

6.

(a) We need to check the properties of an inner product. (Note that since V is a real vector space, we can ignore the conjugation and just show it is symmetric.)

- [I1]: We have

$$\begin{aligned}\langle (a_1 + ra_2, b_1 + rb_2), (c,d)\rangle &= 3(a_1 + ra_2)c + (a_1 + ra_2)d + (b_1 + rb_2)c + (b_1 + rb_2)d \\ &= [3a_1c + a_1d + b_1c + b_1d] + r[3a_2c + a_2d + b_2c + b_2d] \\ &= \langle (a_1, b_1), (c,d)\rangle + \langle (a_2, b_2), (c,d)\rangle .\end{aligned}$$

- [I2]: We have $\langle (c,d), (a,b)\rangle = 3ca + cb + da + db = 3ac + ad + bc + bd = \langle (a,b), (c,d)\rangle$.
- [I3]: We have $\langle (a,b), (a,b)\rangle = 3a^2 + 2ab + b^2 = 2a^2 + (a+b)^2$, which is a sum of squares hence always nonnegative. Furthermore, it is only zero when $a = a + b = 0$, meaning that $(a,b) = (0,0)$.

(b) This is the square of the Cauchy-Schwarz inequality for the inner product in part (a). Explicitly, Cauchy-Schwarz says that $|\langle \mathbf{v}, \mathbf{w}\rangle| \le ||\mathbf{v}|| \cdot ||\mathbf{w}||$. Squaring, and observing that $\langle \mathbf{v}, \mathbf{w}\rangle$ is real, produces $\langle \mathbf{v}, \mathbf{w}\rangle^2 \le \langle \mathbf{v}, \mathbf{v}\rangle \langle \mathbf{w}, \mathbf{w}\rangle$. Now just take $\mathbf{v} = (a,b)$ and $\mathbf{w} = (c,d)$ using the inner product from part (a): we immediately obtain $(3ac + ad + bc + bd)^2 \le (3a^2 + 2ab + b^2)(3c^2 + 2cd + d^2)$.

7.

(a) Note that because $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ is orthonormal, $\langle \mathbf{u}_i, \mathbf{u}_j \rangle$ is 0 when $i \neq j$ and 1 when $i = j$. Expanding out $\langle 2\mathbf{u}_1 - \mathbf{u}_2 + 4\mathbf{u}_3, \mathbf{u}_1 + 2\mathbf{u}_2 + 2\mathbf{u}_3 \rangle$, we obtain

$$2\langle \mathbf{u}_1, \mathbf{u}_1 \rangle + 4\langle \mathbf{u}_1, \mathbf{u}_2 \rangle + 8\langle \mathbf{u}_1, \mathbf{u}_3 \rangle - \langle \mathbf{u}_2, \mathbf{u}_1 \rangle - 2\langle \mathbf{u}_2, \mathbf{u}_2 \rangle - 2\langle \mathbf{u}_2, \mathbf{u}_3 \rangle + 4\langle \mathbf{u}_3, \mathbf{u}_1 \rangle + 8\langle \mathbf{u}_2, \mathbf{u}_3 \rangle + 8\langle \mathbf{u}_3, \mathbf{u}_3 \rangle$$

and the only nonzero terms are $2\langle \mathbf{u}_1, \mathbf{u}_1 \rangle - 2\langle \mathbf{u}_2, \mathbf{u}_2 \rangle + 8\langle \mathbf{u}_3, \mathbf{u}_3 \rangle = 2 - 2 + 8 = \boxed{8}$.
More generally, $\langle a_1\mathbf{u}_1 + \cdots + a_n\mathbf{u}_n, b_1\mathbf{u}_1 + \cdots + b_n\mathbf{u}_n \rangle = a_1b_1 + \cdots + a_nb_n$, as proven on homework 7.

(b) Notice that $||\mathbf{v} + \mathbf{w}||^2 = \langle \mathbf{v} + \mathbf{w}, \mathbf{v} + \mathbf{w} \rangle = \langle \mathbf{v}, \mathbf{v} \rangle + \langle \mathbf{v}, \mathbf{w} \rangle + \langle \mathbf{w}, \mathbf{v} \rangle + \langle \mathbf{w}, \mathbf{w} \rangle = a + 2\langle \mathbf{v}, \mathbf{w} \rangle + b$. So $||\mathbf{v} + \mathbf{w}||^2$ will be equal to $a + b$ precisely when $2\langle \mathbf{v}, \mathbf{w} \rangle = 0$, which is to say, precisely when $\mathbf{v}$ and $\mathbf{w}$ are orthogonal.
