MTH 235: Linear Algebra

Midterm 1

February 28, 2017

NAME (please print legibly): ______________________________

Your University ID Number: ______________________________

Circle your instructor:

Evan Dummit   Carl M$^{\text{c}}$Tague

- **The presence of electronic devices (including calculators), books, or formula cards/sheets at this exam is strictly forbidden.**
- **Show all work and justify all answers. You MUST provide complete, clear answers for each problem, and justify each step in your proofs.**
- **You may appeal to any theorems, propositions, etc. covered at any point in the course, but please make clear what results you are using.**
- **In problems with multiple parts, you may use the results of previous parts in later parts, EVEN if you did not solve the earlier parts correctly.**
- **You are responsible for checking that this exam has all 7 pages.**

Pledge of Honesty

I affirm that I will not give or receive any unauthorized help on this exam, and that all work will be my own.

Signature: ______________________________

QUESTION	VALUE	SCORE
1	8	
2	8	
3	25	
4	14	
5	16	
6	15	
7	14	
TOTAL	100	

1. (8 points) Prove that

$$1 \cdot 2 + 2 \cdot 3 + \cdots + (n-1) \cdot n = \frac{n^3 - n}{3}$$

for any integer $n \geq 2$.

2. (8 points) For a complex number z, prove that $\overline{z} = -z$ if and only if $z = ri$ for some real number r.

3. (25 points) For each of the following, circle the correct response (there is no partial credit or penalty for wrong answers, and no work is required):

True **False** If W_1 and W_2 are subspaces of V, then the set of vectors in both W_1 and W_2 is a subspace of V.

True **False** The set of polynomials $p(x)$ with $p(1) = 0$ is a subspace of the space of real-valued functions.

True **False** The set $\{1 + t^2, t - t^2 + t^3, 3 - 2t + t^3\}$ spans $P_3(\mathbb{R})$.

True **False** The dimension of a vector space is always positive.

True **False** If $\dim(V) = 3$, then every set of 3 or more vectors spans V.

True **False** If $\dim(V) = 3$, then no basis of V can have 2 elements.

True **False** The vectors $\langle 1, 1, 4\rangle$, $\langle 2, 0, 2\rangle$, $\langle 1, 3, 2\rangle$, $\langle 4, 7, 1\rangle$ are linearly independent.

True **False** For any vector $\mathbf{x}$ in an inner product space, $\langle 3\mathbf{x}, 2\mathbf{x}\rangle \geq 0$ is always true.

True **False** In any inner product space, $||\mathbf{v} + \mathbf{w}|| \leq ||\mathbf{v}|| + ||\mathbf{w}||$ for any vectors $\mathbf{v}$, $\mathbf{w}$.

True **False** The vectors $\frac{1}{\sqrt{2}}\langle 1, 0, 1\rangle$, $\langle 0, 1, 0\rangle$, $\frac{1}{\sqrt{2}}\langle 1, 0, -1\rangle$ are an orthonormal basis for $\mathbb{R}^3$ (with the standard dot product).

4. (14 points) Let $V = \mathbb{R}^5$ and let S be the set of vectors $\mathbf{v} = \langle x_1, x_2, x_3, x_4, x_5 \rangle$ such that $x_5 = x_1 + x_2$ and $x_3 = x_4$.

(a) Prove that S is a subspace of V.

(b) Find a basis for S and the dimension of S.

5. (16 points) In a vector space V, let

$$S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_{n-1}, \mathbf{v}_n\}$$
$$T = \{\mathbf{v}_1 - \mathbf{v}_2, \mathbf{v}_2 - \mathbf{v}_3, \ldots, \mathbf{v}_{n-1} - \mathbf{v}_n, \mathbf{v}_n\}.$$

(a) Suppose S spans V. Prove that T also spans V.

(b) Suppose S is linearly independent. Prove that T is also linearly independent.

(c) Suppose S is a basis for V. Prove that T is a basis for V.

6. (15 points) Let $V = \mathbb{R}^2$.

(a) Show that the pairing $\langle (a,b),(c,d)\rangle = 3ac + ad + bc + bd$ is an inner product on V.

(b) Prove that, for any real numbers a, b, c, d, it is true that

$$(3ac + ad + bc + bd)^2 \leq (3a^2 + 2ab + b^2)(3c^2 + 2cd + d^2).$$

7. (14 points) Let V be a real inner product space.

(a) Suppose that $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ is an orthonormal set in V. Find $\langle 2\mathbf{u}_1 - \mathbf{u}_2 + 4\mathbf{u}_3,\ \mathbf{u}_1 + 2\mathbf{u}_2 + 2\mathbf{u}_3\rangle$.

(b) Suppose that $||\mathbf{v}|| = \sqrt{a}$ and $||\mathbf{w}|| = \sqrt{b}$. Show that $||\mathbf{v} + \mathbf{w}|| = \sqrt{a+b}$ if and only if $\mathbf{v}$ and $\mathbf{w}$ are orthogonal.