MATH 235: Midterm Practice Assignment - Sections 1.1-2.5
For Practice Only – Do NOT hand-in

(P1) True of False?? For each of the following, decide if the statement is true or false. Fully justify for your answer.

(a) In a finite dimensional vector space all bases have the same number of vectors.

(b) If V is a vector space with $1 \leq \dim(V) = n < \infty$, then V has exactly one subspace of dimension 0 and exactly one subspace of dimension n.

(c) In a vector space V with $\dim(V) = n < \infty$, a set of k vectors is linearly independent if and only if $k \leq n$.

(d) In a vector space V with $\dim(V) = n < \infty$, a set of k vectors is generates V if and only if $k \geq n$.

(e) For $A \in M_{n\times n}(\mathbb{R})$, L_A is onto if and only if $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.

(f) For a square matrix $A \in M_{n\times n}(\mathbb{R})$, the system $A\mathbf{x} = \mathbf{0}$ only has the trivial solution if and only if the column vectors of A are a basis for $\mathbb{R}^n$.

(g) If A is an $m \times n$ matrix with $\text{rank}(L_A) = m$, then the system $A\mathbf{x} = \mathbf{b}$ always has a solution.

(h) If S_1 is linearly dependent and $S_1 \subset S_2$, then S_2 is linearly dependent.

(P2) True of False?? For each of the following, decide if the statement is true or false. Fully justify for your answer. Throughout, assume V is a vector space with $\dim(V) = n < \infty$ and $T : V \to W$ is linear.

(a) If $\{\mathbf{v}_1, \cdots, \mathbf{v}_k\}$ is a basis for $N(T)$, then $k \leq n$.

(b) If $\{\mathbf{w}_1, \cdots, \mathbf{w}_k\}$ is a basis for $R(T)$, then $k \leq n$.

(c) If $\{\mathbf{x}_1, \cdots, \mathbf{x}_m\}$ generates V, then $\{T(\mathbf{x}_1), \cdots, T(\mathbf{x}_m)\}$ generates W.

(d) If $\{\mathbf{v}_1, \cdots, \mathbf{v}_k\}$ is a basis for $N(T)$ and $\{\mathbf{v}_1, \cdots, \mathbf{v}_n\}$ is a basis for V, then $\{T(\mathbf{v}_{k+1}), \cdots, T(\mathbf{v}_n)\}$ is a basis for $R(T)$.

(P3) Let $T : V \to W$ and $U : W \to Z$ be linear where V, W, Z are finite dimensional vector spaces.

(a) Prove that $N(T)$ is a subspace of $N(UT)$.

(b) Prove that if U is 1-1, then $N(T) = N(UT)$.

(c) Prove that if U is 1-1, then $\text{rank}(T) = \text{rank}(UT)$. Is it necessarily true that $R(T) = R(UT)$?

(P4) Find 2×2 matrices A and B such that $AB = O$ but $BA \neq O$. Similarly, find linear transformations $U, T : F^2 \to F^2$ such that $UT = T_0$ (the zero transformation) but $TU \neq T_0$.

(P5) Answer each part below.

(a) Find a matrix A such that

$$A\begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

and

$$A\begin{pmatrix} 3 \\ 2 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Is there another matrix satisfying this or is the matrix you found unique?

(b) Find a matrix B satisfying

$$B\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

and

$$B\begin{pmatrix} 2 \\ 2 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \end{pmatrix}$$

Is there another matrix satisfying this or is the matrix you found unique?

(P6) (a) Let V and W be vector spaces with $\dim(V) = \dim(W) < \infty$ and $\beta = \{\vec{v}_1, \vec{v}_2, \cdots, \vec{v}_n\}$, $\gamma = \{\vec{u}_1, \vec{u}_2, \cdots, \vec{u}_n\}$ be bases on V and W respectively. Construct an isomorphism from V onto W in the most reasonable way possible in the context of this problem. [Hint: see the proof of Theorem 2.19 - you do not have to prove your map is an isomorphism.]

(b) Construct an isomorphism from $\mathbb{R}^2$ to $\mathbb{R}^2$ which is not the identity transformation on $\mathbb{R}^2$.

(P7) In each part below, decide if the given vector spaces V and W are isomorphic. If so, construct an isomorphism $T : V \to W$. If not, explain in just a few words why they are not isomorphic.

(a) $V = \mathbb{R}^2$ and $W = \mathbb{R}^3$.

(b) $V = \mathbb{R}^2$ and $W = M_{2\times 2}(\mathbb{R})$.

(c) $V = P_5(\mathbb{R})$ and $W = M_{3\times 2}(\mathbb{R})$.

(P8) True of False?? For each of the following, decide if the statement is true or false. Provide brief justification for your answer (you do not have to give full-blown proofs, brief explanations will suffice). In each part, V and W are finite dimensional vector spaces over F with ordered bases β and γ respectively, $T : V \to W$ is linear, and $A \in M_{n\times n}(F)$.

(a) $T^{-1} : W \to V$ exists if and only if T is a bijection (i.e. one-to-one and onto).

(b) I_V (the identity transformation on V) is invertible and $I_V^{-1} = I_V$.

(c) The $n \times n$ zero matrix is invertible.

(d) If $\left([T]_\beta^\gamma\right)^{-1}$ exists, then $\left([T]_\beta^\gamma\right)^{-1} = [T^{-1}]_\beta^\gamma$.

(e) A is invertible if and only if L_A is invertible, and in this case $L_A^{-1} = L_{A^{-1}}$.

(f) A is invertible if and only if L_A is an isomorphism from F^n onto itself.

(P9) Let $T : P_1(\mathbb{R}) \to \mathbb{R}^2$ be the linear map defined by $T(a + bx) = (-b, a + 2b)$.

(a) Find $[T]_\beta^\gamma$ where β, γ are the standard ordered bases on P_1 and $\mathbb{R}^2$ respectively.

(b) Find $\left([T]_\beta^\gamma\right)^{-1}$.

(c) Use part (b) to find a formula for $T^{-1}(c, d)$.

(P10) True of False?? For each of the following, decide if the statement is true or false. Provide brief justification for your answer (you do not have to give full-blown proofs, brief explanations will suffice). In each part, T is a linear operator on a finite dimensional vector space V, $\beta = \{\mathbf{x_1}, \mathbf{x_2}, \cdots, \mathbf{x_n}\}$ and $\beta' = \{\mathbf{x'_1}, \mathbf{x'_2}, \cdots, \mathbf{x'_n}\}$ are both bases for V, and Q is the change of coordinates matrix that changes β' coordinates into β coordinates.

(a) $Q = [I_V]_{\beta'}^\beta$.

(b) The jth column of Q is $[x_j]_{\beta'}$.

(c) Q^{-1} necessarily exists and it is the change of coordinates matrix from β to β' coordinates.

(d) $[T]_\beta = Q[T]_{\beta'}Q^{-1}$.

(e) If γ is another basis for V and P is the change of coordinates matrix from β to γ coordinates, then PQ is the change of coordinates matrix from β' to γ coordinates.

(P11) Let β be the standard ordered basis for $\mathbb{R}^2$, $\alpha = \left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \end{pmatrix} \right\}$, and

$\gamma = \left\{ \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 3 \end{pmatrix} \right\}$.

(a) Find Q_1 and Q_2, where Q_1 is the change of coordinates matrix from α to β and Q_2 is the change of coordinates matrix from γ to β.

(b) Use your answer from part (a) to find the change of coordinates matrices from β to α and β to γ.

(c) Use your answers from parts (a) and (b) to find the change of coordinates matrix from α to γ.

(d) For the matrix

$$A = \begin{pmatrix} 1 & -2 \\ 3 & 1 \end{pmatrix}$$

find $[L_A]_\beta$, $[L_A]_\alpha$, and $[L_A]_\gamma$. Note: it will help to utilize the matrices you found in the previous parts of this problem.

(P12) Let β be the standard ordered basis on P_1, $\beta' = \{1 + 2x, 1 - x\}$.

(a) Use a change of coordinates matrix to find $[p_1]_{\beta'}$, $[p_2]_{\beta'}$, where $p_1 = 3 - 2x$, $p_2 = 1 + 4x$.

(b) Let T be the linear operator on P_1 defined by $T(p) = p'$, the derivative of p. Use theorem 2.23 to find $[T]_{\beta'}$.