MATH 235: Homework 4 - Sections 2.1-2.3
Due Saturday, 7/13 at 1pm on Gradescope

(P1) Suppose that $T : \mathbb{R}^2 \to \mathbb{R}^2$ is linear and that $T((1,2)) = (3,4)$ and $T((1,3)) = (0,1)$. Find $T((1,0))$. Is T one-to-one? Justify your answer.

(P2) You are given two maps between vector spaces over the same field. For each map

(i) Show that it is linear.

(ii) Decide whether it is one-to-one or not (with justification).

(iii) Decide whether it is onto or not (with justification).

(a) $T : P_3(\mathbb{R}) \to M_{2\times 2}(\mathbb{R})$ defined by $T(p) = \begin{pmatrix} p(0) & p'(0) \\ p''(0) & p'''(0) \end{pmatrix}$.

(b) $T : \mathbb{R}^2 \to \mathbb{R}^3$ defined by $T((a,b)) = (a, b, a+b)$.

[Hint: Use Rank-Nullity]

(P3) Let V and W be vector spaces and $T : V \to W$ be linear.

(a) Show that if V_1 is a subspace of V, then $T(V_1) = \{T(x) \,|\, x \in V_1\}$ is a subspace of W.

(b) Show that if W_1 is a subspace of W, then $\{\mathbf{x} \in V : \; T(\mathbf{x}) \in W_1\}$ is a subspace of V.

(P4) In each of the following parts you are given vector spaces V and W, the ordered basis for these vector spaces β, γ and a linear map $T : V \to W$. Write down $[T]_\beta^\gamma$.

(a) $V = \mathbb{R}^3$, $\beta = \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$, $W = P_2(\mathbb{R})$,$\gamma = \{1, t, t^2\}$ and $T((a,b,c)) = a + ct^2$.

(b) $V = M_{2\times 2}(\mathbb{R})$, $\beta = \{\mathbf{E}^{11}, \mathbf{E}^{12}, \mathbf{E}^{21}, \mathbf{E}^{22}\}$, $W = \mathbb{R}^3$, $\gamma = \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ and

$$T(\begin{pmatrix} a & b \\ c & d \end{pmatrix}) = (a+d, b+c, a+b+c+d).$$

$\mathbf{E}^{ij}$ is the matrix whose (i,j) entry is a 1 and all other entries are 0's.

(c) $V = \mathbb{R}^3$, $\beta = \{\mathbf{e}_1 + \mathbf{e}_2, \mathbf{e}_2, \mathbf{e}_3 - \mathbf{e}_1\}$, $W = \mathbb{R}^2$, $\gamma = \{\mathbf{e}_1, 2\mathbf{e}_2\}$ and $T((a, b, c)) = (a, b + c)$.

(d) $V = P_1(\mathbb{R})$, $\beta = \{1 + t, 1 - t\}$, $W = \mathbb{R}^2$, $\gamma = \{\mathbf{e}_1 + \mathbf{e}_2, \mathbf{e}_1 - \mathbf{e}_2\}$ and $T(p) = (p(0), p'(0))$.

(P5) Let V and W be finite dimensional vector spaces over the same field F and $T : V \to W$ be a one-to-one linear map. Prove each of the following.

(a) A set $L \subseteq V$ is linearly independent if and only if $T(L) \subseteq W$ is linearly independent.

(b) Suppose further that $\dim(V) = \dim(W)$. Then a set $\beta \subseteq V$ is a basis for V if and only if $T(\beta)$ is a basis for W.

(c) Deduce that if $\dim(V) = \dim(W)$, then there exist ordered bases β and γ for V and W respectively such that $[T]_\beta^\gamma$ is the identity matrix as follows: for any ordered basis β of V choose one by one the elements of a basis γ for W so that the ith column of $[T]_\beta^\gamma$ is $\mathbf{e}_i$. [Hint:Visualize by drawing a picture]

(P6) Let V, W, and Z be vector spaces, and let $T : V \to W$ and $U : W \to Z$ be linear.

(a) Prove that if UT is one-to-one, then T is one-to-one. Must U also be one-to-one?

(b) Prove that if UT is onto, the U is onto. Must T also be onto?

(c) Prove that if U and T are bijections (one-to-one and onto), then UT is also.

(P7) Let V be a vector space, and let $T : V \to V$ be linear. Prove that $T^2 = T_0$ if and only if $R(T) \subseteq N(T)$.

(P8) Let $T : P_2(\mathbb{R}) \to P_2(\mathbb{R})$ and $U : P_2 \to \mathbb{R}^3$ be defined by:

$$T(f) = (2 - x^2)f'' + 2xf' + 3f, \qquad U(a + bx + cx^2) = (a + c, a - b, b)$$

Let β and γ be the standard bases on P_2 and $\mathbb{R}^3$ respectively.

(a) Compute $[U]_\beta^\gamma, [T]_\beta$, and $[UT]_\beta^\gamma$ directly. Then use Theorem 2.11 to verify your result.

(b) Let $f(x) = -4 + x - 3x^2$. Compute $[f]_\beta$ and $[U(f(x))]_\gamma$ directly. Then use $[U]_\beta^\gamma$ from (a) and Theorem 2.14 to verify your result.