MATH 235: Homework 3
Due Wednesday, 7/10 at 1pm on Gradescope

(P1) $W = \{(x_1, \ldots, x_4) \in \mathbb{R}^4 : x_1 + \cdots + x_4 = 0$ is a subspace of $\mathbb{R}^4$. The set S below spans W (you do not have to prove this):

$$S = \left\{ \mathbf{v}_1 = \begin{pmatrix} 0 \\ 2 \\ 1 \\ -3 \end{pmatrix}, \mathbf{v}_2 = \begin{pmatrix} -2 \\ 1 \\ 3 \\ -2 \end{pmatrix}, \mathbf{v}_3 = \begin{pmatrix} 2 \\ 3 \\ -1 \\ -4 \end{pmatrix}, \mathbf{v}_4 = \begin{pmatrix} 2 \\ 13 \\ 4 \\ -19 \end{pmatrix}, \mathbf{v}_5 = \begin{pmatrix} -3 \\ 1 \\ 1 \\ 1 \end{pmatrix} \right\}.$$

Determine a subset of S that is a basis for W (justify briefly).

(P2) You are given a set S in a vector space V. Determine whether S is a basis and justify your answers.

(i) $V = \mathbb{R}^3$ and

$$S = \{\ (1, -1, 2),\ (1, 4, -1),\ (5, 5, 4)\ \}.$$

(ii) $V = P_3(\mathbb{R})$ and $S = \{p_1, p_2, p_3, p_4, p_5\}$, where

$$\begin{aligned} p_1 &= 1 - t + t^2 - t^3 \\ p_2 &= 1 + 2t + 3t^2 + 4t^3 \\ p_3 &= t - t^2 - t^3 \\ p_4 &= t^2 + t^3 \\ p_5 &= 1 + 6t + 4t^2 + 8t^3 \end{aligned}$$

(iii) $V = M_2(\mathbb{R})$ and

$$S = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 2 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 2 & 3 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \right\}.$$

(P3) Show that $\beta = \{(1, -1, 0),\ (0, 1, 1),\ (-1, 0, 1)\}$ is a basis for $\mathbb{R}^3$. Find the unique representation of the vector (a, b, c) as a linear combination of the elements of β.

(P4) You are given a vector space V and a subspace $W \subset V$. Write down the dimension of W (with justification).

(i) $V = M_n(\mathbb{R})$ and W is the space of skew-symmetric matrices.

(ii) $V = P_4(\mathbb{R})$ and W is the space of all polynomials $p(x)$ whose second derivative is zero for all $x \in (-\infty, \infty)$.

(P5) Let $W_1 = \{\mathbf{u} \in \mathbb{R}^5 : u_1 + u_3 + u_4 = 0,\ 2u_1 + 2u_2 + u_5 = 0\}$ and $W_2 = \{\mathbf{u} \in \mathbb{R}^5 : u_1 + u_5 = 0,\ u_2 = u_3 = u_4\}$. These are subspaces of $\mathbb{R}^n$ (you do not have to prove this).

(a) Find a basis for $W_1 \cap W_2$. (Justify briefly your reasoning)

(b) Extend your basis from part (a) to a basis for W_1. (Justify briefly your reasoning)

(P6) Let W_1 and W_2 be subspaces of a finite dimensional vector space. Determine necessary and sufficient conditions on W_1 and W_2 so that $\dim(W_1 \cap W_2) = \dim(W_1)$.

(P7) Let $\mathbf{u}_1, \ldots, \mathbf{u}_k, \mathbf{v}$ be vectors in a vector space V. Find necessary and sufficient conditions for $\dim(\text{span}(\{\mathbf{u}_1, \ldots, \mathbf{u}_k\})) = \dim(\text{span}(\{\mathbf{u}_1, \ldots, \mathbf{u}_k, \mathbf{v}\}))$.

(P8) Let W_1 and W_2 be subspaces of a finite dimensional vector space V with basis β, γ respectively. Suppose that $V = W_1 \oplus W_2$. Show that $\beta \cap \gamma = \emptyset$ and that $\beta \cup \gamma$ is a basis for V.