MATH 235: Homework 2
Due Monday, 7/8 at 1pm on Gradescope

(P1) Read the bottom of pg. 22 in the book where the sum of two non-empty subsets is defined. Let W_1 and W_2 be subspaces of a vector space V. Prove that $W_1 + W_2$ is the minimal subspace of V that contains both W_1 and W_2. That is, prove it is a subspace containing both W_1 and W_2 and that any subspace of V that contains both W_1 and W_2 must also contain $W_1 + W_2$.

(P2) Read the bottom of pg. 22 in the book where the direct sum of two vector spaces is defined. Let

$$W_1 = \{M \in M_2(\mathbb{R}) : M_{11} = 0\} \quad \text{and} \quad W_2 = \{M \in M_2(\mathbb{R}) : M_{12} = M_{21} = M_{22} = 0\}.$$

Prove that W_1 and W_2 are subspaces of $M_2(\mathbb{R})$ and that $M_2(\mathbb{R}) = W_1 \oplus W_2$.

(P3) Let W_1 and W_2 be subspaces of a vector space V. Prove that $V = W_1 \oplus W_2$ if and only if each vector in V can be *uniquely* written as $x_1 + x_2$, where $x_1 \in W_1$ and $x_2 \in W_2$.

(P4) You are given a set S in a vector space V. Determine whether S spans V. Justify your answers.

(i) $V = \mathbb{R}^3$ and

$$S = \{\ (1, 2, -1),\ (0, 1, 1),\ (2, 1, -5),\ (-3, 1, 10)\ \}$$

(ii) $V = M_{2\times 2}(\mathbb{R})$ and

$$S = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \right\}$$

(iii) $V = C^0[-1, 1]$ (the vector space of continuous functions on $[-1, 1]$) and $S = \{1, t, t^2, t^3, \dots\}$.

(P5) Show that the span of $\left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right\}$ is the set of all symmetric 2×2 matrices.

(P6) Let S be a set in a vector space V and $\mathbf{v}$ any vector. Prove that $\text{span}(S) = \text{span}(S \cup \{\mathbf{v}\})$ if and only if $\mathbf{v} \in \text{span}(S)$.

(P7) Prove that a set S in a vector space V is linearly dependent if and only if there exists a proper subset $S' \subsetneq S$ with the same span as S.

(P8) You are given a set S in a vector space V. Determine whether S is linearly dependent using exclusively methods developed in this course and justify your answers.

(i) $V = \mathbb{R}^3$ and

$$S = \{\ (1, 2, -1),\ (2, -3, 1),\ (2, 3, -5),\ \}$$

(ii) $V = P_3(\mathbb{R})$ and

$$S = \{\ 1,\ 1 + 2t + t^2,\ 1 - 2t + t^3,\ t^2 + t^3\ \}$$

(iii) $V = \mathcal{F}(\mathbb{R}, \mathbb{R})$ and $S = \{t, e^t, \sin(t)\}$.

(P9) Let $\mathbf{u}$ and $\mathbf{v}$ be vectors in a vector space V. Show that $\{\mathbf{u}, \mathbf{v}\}$ is linearly independent if and only if $\{\mathbf{u} + \mathbf{v}, \mathbf{u} - \mathbf{v}\}$ is linearly independent.

(P10) Let $S = \{\mathbf{u}_1, \ldots, \mathbf{u}_n\}$ be a finite set of vectors in a vector space V. Prove that S is linearly dependent if and only if $\mathbf{u}_1 = 0$ or $\mathbf{u}_{k+1} \in \operatorname{span}(\{\mathbf{u}_1, \ldots, \mathbf{u}_k\})$ for some $1 \leq k < n$.