MATH 235: Homework 1

Due Wednesday, 7/3 at 1pm on Gradescope

(P1) You are given a set V and two operations $\boxplus$ and $\boxdot$. Explain whether $(V, \boxplus, \boxdot)$ is a real vector space (i.e. a vector space over $\mathbb{R}$) or not by providing a proof of the axioms or a counter example to one of the axioms.

(i) $V = \mathbb{R}^2$ and for $\mathbf{u} = (u_1, u_2), \mathbf{v} = (v_1, v_2) \in \mathbb{R}^2$ and $\lambda \in \mathbb{R}$:

$$\mathbf{u} \boxplus \mathbf{v} = (u_1 + v_2, u_2 + v_1) \text{ and } \lambda \boxdot u = (\lambda u_1, \lambda u_2).$$

(ii) $V = \mathbb{R}^2$ and for $\mathbf{u} = (u_1, u_2), \mathbf{v} = (v_1, v_2) \in \mathbb{R}^2$ and $\lambda \in \mathbb{R}$:

$$\mathbf{u} \boxplus \mathbf{v} = (u_1 + v_1, u_2 + v_2) \text{ and } \lambda \boxdot u = (\lambda u_1, u_2).$$

(P2) You are given a subset S of a well known vector space V. Is S is a subspace of V? If so, prove it. If not, provide a counterexample. The notation is identical to that of the book.

(i) $V = \mathcal{F}(\mathbb{R}, \mathbb{R})$ the set of real valued functions on $\mathbb{R}$ and

$$S = \{f \in V : \lim_{t \to \infty} f(t) = 0\}.$$

(ii) V is the set of real sequences and

$$S = \{ \{a_n\} \in V : a_1 + 3a_2 \leq 1 \}.$$

Note: for notation, etc. see example 5 from section 1.2.

(iii) $V = C^1(\mathbb{R})$ the set of continuously differentiable functions on $\mathbb{R}$ and

$$S = \{f(t) \in V : (f'(t))^2 + (f(t))^2 = 0\}.$$

(P3) Prove that the set

$$S = \{(x, y, z) \in \mathbb{R}^3 : x + y + z = b\}.$$

is a subspace of $\mathbb{R}^3$ if and only if $b = 0$.

(P4) A function $f : \mathbb{R} \to \mathbb{R}$ is called additive if

$$f(x + y) = f(x) + f(y) \text{ for all } x, y \in \mathbb{R}.$$

Is the set of all additive functions a subspace of $\mathcal{F}(\mathbb{R}, \mathbb{R})$? Give a proof or counterexample.

(P5) Let W_1 and W_2 be subspaces of a vector space V. Prove that $W_1 \cup W_2$ is a subspace of V if and only if $W_1 \subseteq W_2$ or $W_2 \subseteq W_1$.