MATH 235

Final ANSWERS

May 5, 2015

1. (10 points)

Fix positive integers m, n and consider the vector space $\mathbf{V}$ of all $m \times n$ matrices with entries in the real numbers $\mathbf{R}$.

(a) Find the dimension of $\mathbf{V}$ and prove your answer. Please carry out all the steps of your proof.

(b) Let $\mathbf{P}$ be the subset of $\mathbf{V}$ consisting of $m \times n$ matrices each of whose row sum is 1. Prove or disprove: $\mathbf{P}$ is a subspace of $\mathbf{V}$.

(c) Assume $m \geq 2$ and $n \geq 2$. Find a subspace of $\mathbf{V}$ of dimension 2. Please explain your answer, but you don't have to give a proof.

Answer:

(a) $\mathbf{V}$ has dimension mn. To prove this statement, it is enough to exhibit a basis β of $\mathbf{V}$ with mn elements. For $1 \leq i \leq m$ and $1 \leq j \leq n$, let $\mathbf{e}_{ij}$ denote the matrix in $\mathbf{V}$ with a 1 in the (i, j) position and zeros elsewhere. Note that there are mn such elements $\mathbf{e}_{ij}$, and let β be the set of these mn matrices. If β forms a basis of $\mathbf{V}$, then $\mathbf{V}$ has dimension mn and we are done.

First we show that the span of β is $\mathbf{V}$. Indeed, if $A \in \mathbf{V}$, then

$$A = \sum_{i=1}^{m} \sum_{j=1}^{n} A_{ij} \mathbf{e}_{ij}$$

so β spans $\mathbf{V}$.

Next we show that β is a linearly independent set, which would finish the proof that β is a basis for $\mathbf{V}$ and hence the dimension of $\mathbf{V}$ is mn. Suppose that $0 = A = \sum_{i=1}^{m} \sum_{j=1}^{n} a_{ij} \mathbf{e}_{ij}$ for scalars a_{ij}. By the definition of $\mathbf{e}_{ij}$, we conclude that $a_{ij} = A_{ij}$. But a matrix $A = 0$ if and only if all of its entries $A_{ij} = a_{ij}$ are equal to 0. This shows that β is a linearly independent set, and hence the dimension of $\mathbf{V}$ is mn.

(b) $\mathbf{P}$ cannot be a subspace of $\mathbf{V}$. Indeed, let $A \in \mathbf{P}$. By the definition of $\mathbf{P}$, each of the rows of A sums to 1. Now consider $B = 2A$, and note that each of the rows of B sums to 2. Therefore, $B \notin \mathbf{P}$, and so $\mathbf{P}$ is not closed under scalar multiplication. Therefore, $\mathbf{P}$ is not a vector space, and hence not a subspace of $\mathbf{V}$.

(c) It suffices to find a linearly independent subset of $\beta \subset \mathbf{V}$ with 2 nonzero elements, since then the span of β would be a two dimensional subspace of $\mathbf{V}$. There are many such choices. For example, using the notation from part (a) above, we could choose $\beta = \{\mathbf{e}_{11}, \mathbf{e}_{12}\}$.

2. (10 points)

$P_2(\mathbf{R})$ is the real vector space of real polynomials of degree at most 2. Let W be the following subset of $P_2(\mathbf{R})$:

$$W = \{f \in P_2(\mathbf{R}) \,|\, f(2) = f(1)\}.$$

(a) Prove that W is a vector subspace of $P_2(\mathbf{R})$.

(b) Write down a basis for W. You do not need to prove that the set given is a basis, though justification of how you found it must be given.

(c) W is isomorphic to $\mathbf{R}^d$ for what value of d? Justify your answer.

Answer:

(a) W is a subspace because it satisfies the three standard properties.

1. The zero polynomial $O(x)$ belongs in W: $O(2) = 0 = O(1)$.
2. W is closed under addition: let f and g be two polynomials in W, their sum $(f + g)$ also belongs to W because
$$(f + g)(2) = f(2) + g(2) = f(1) + g(1) = (f + g)(1).$$
3. W is closed under scalar multiplication: let f be a polynomial in W and λ a real number, their scalar product (λf) also belongs to W because
$$(\lambda f)(2) = \lambda f(2) = \lambda f(1) = (\lambda f)(1).$$

(b) Let $f(t) = at^2 + bt + c \in W$. Then

$$4a + 2b + c = f(2) = f(1) = a + b + c.$$

Therefore, $3a + b = 0$.

This means that c is a free variable and a, b are related by $b = -3a$. Therefore,

$$f(t) = a(t^2 - 3t) + c$$

and a basis for W is $\{t^2 - 3t, 1\}$.

(c) The dimension of W is 2 and so W is isomorphic to $\mathbf{R}^2$ because all 2-dimensional real vector spaces are isomorphic to $\mathbf{R}^2$.

3. (10 points)

Let $\mathbf{V}$ denote the linear span of the following functions from $\mathbf{R}$ to $\mathbf{R}$: $e^{-2x}, 1, e^{2x}$. Also suppose that these functions form an ordered basis β for $\mathbf{V}$. Let $T : \mathbf{V} \to \mathbf{V}$ be the linear transformation defined by $(Tf)(x) = f(-x)$, and let $D : \mathbf{V} \to \mathbf{V}$ be the linear transformation defined by $(Df)(x) = \frac{df(x)}{dx}$.

For the following questions, you must show your calculations, but you need not give a proof.

(a) Find the matrix $[T]_\beta$.

(b) Find the matrix $[D]_\beta$.

(c) Find the matrix $[TD]_\beta$.

Answer:

(a) We find that T maps $e^{-2x}, 1, e^{2x}$ to $e^{2x}, 1, e^{-2x}$ respectively. Therefore

$$[T]_\beta = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

(b) We find that D maps $e^{-2x}, 1, e^{2x}$ to $-2e^{-2x}, 0, 2e^{2x}$ respectively. Therefore

$$[D]_\beta = \begin{pmatrix} -2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 2 \end{pmatrix}.$$

(c) Using parts (a) and (b), we find

$$[TD]_\beta = [T]_\beta[D]_\beta = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} -2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 2 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 2 \\ 0 & 0 & 0 \\ -2 & 0 & 0 \end{pmatrix}.$$

4. (10 points)

Let $T : \mathbf{R}^2 \to \mathbf{R}^2$ be linear and suppose $T^2 \neq 0$, where $T^2 = T \circ T$ and 0 denotes the zero map.

(a) Show that $1 \leq \text{rank}(T^2) \leq \text{rank}(T)$.

(b) By considering the possible values of $\text{rank}(T)$ separately, deduce that $R(T) = R(T^2)$, where, say, $R(T)$ is the range of T.

Answer:

(a) $T^2 \neq 0$ implies $\text{rank}(T) \geq 1$. Moreover,

$$R(T^2) = T[T[\mathbf{R}^2]] \subseteq T[\mathbf{R}^2] = R(T),$$

which implies $\text{rank}(T^2) \leq \text{rank}(T)$. Putting the two inequalities together gives

$$1 \leq \text{rank}(T^2) \leq \text{rank}(T).$$

(b) The possible values of $\text{rank}(T)$ are 1 and 2. Let us consider them separately.

- If $\text{rank}(T) = 1$, then $\text{rank}(T^2)$ is bounded by above and below by 1 and therefore $\text{rank}(T) = 1$. It follows that $R(T^2)$ is a subspace of $R(T)$. These two spaces have the same dimension. Therefore, $R(T) = R(T^2)$.
- If $\text{rank}(T) = 2$, then T is onto. T^2 is also onto because

  $$T^2[\mathbf{R}^2] = T[T[\mathbf{R}^2]] = T[\mathbf{R}^2] = \mathbf{R}^2.$$

  Therefore, $R(T) = \mathbf{R}^2 = R(T^2)$.

In either case, $R(T) = R(T^2)$.

5. (10 points)

Use elementary row and/or column operations to find the determinant of

$$A = \begin{pmatrix} 1 & 1 & 2 & 0 \\ 1 & 0 & 1 & 3 \\ 2 & 1 & 1 & 2 \\ 0 & 2 & 1 & 3 \end{pmatrix}.$$

You must use the method of row and column operations to get any credit for this problem. It's also the easiest way.

Answer:

We can solve the problem using just row operations. First we add multiples of the first row to the other rows, in order to clear the first column (except for its first entry). Such row operations will not affect the determinant. We get:

$$\begin{pmatrix} 1 & 1 & 2 & 0 \\ 0 & -1 & -1 & 3 \\ 0 & -1 & -3 & 2 \\ 0 & 2 & 1 & 3 \end{pmatrix}.$$

Next, add multiples of the second row to the third and fourth rows, to get:

$$\begin{pmatrix} 1 & 1 & 2 & 0 \\ 0 & -1 & -1 & 2 \\ 0 & 0 & -2 & -1 \\ 0 & 0 & -1 & 9 \end{pmatrix}.$$

Again, the determinant is unchanged. Finally, subtract 1/2 times the third row from the fourth row to get

$$\begin{pmatrix} 1 & 1 & 2 & 0 \\ 0 & -1 & -1 & 2 \\ 0 & 0 & -2 & 1 \\ 0 & 0 & 0 & 19/2 \end{pmatrix}.$$

Once again, the determinant is unchanged. Now we have an upper triangular matrix whose determinant is the product of the diagonal entries. Thus

$$\det(A) = 1 \cdot (-1) \cdot (-2) \cdot (19/2) = 19.$$

6. (10 points)

Let $T : \mathbf{R}^2 \to \mathbf{R}^2$ be a linear map and β the following basis for $\mathbf{R}^2$:

$$\beta = \left\{ \begin{pmatrix} 1 \\ 3 \end{pmatrix}, \begin{pmatrix} 2 \\ 4 \end{pmatrix} \right\}.$$

Suppose that T is represented by the following matrix A in β:

$$A := [T]_\beta = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}.$$

(a) Find the nullity of T showing all your work.

(b) Find the matrix representing T in the standard basis for $\mathbf{R}^2$ showing all your work.

Answer:

(a) $\det(A) = -2 \neq 0$ so $\operatorname{rank}(T) = \operatorname{rank}(A) = 2$. By Rank-Nullity,

$$\operatorname{nullity}(T) = \dim(\mathbf{R}^2) - \operatorname{rank}(T) = 2 - 2 = 0.$$

(b) Let α be the standard basis for $\mathbf{R}^2$. The change of coordinates process gives

$$[T]_\alpha = [I_{\mathbf{R}^2}]_\beta^\alpha \, [T]_\beta \, [I_{\mathbf{R}^2}]_\alpha^\beta = [I_{\mathbf{R}^2}]_\beta^\alpha \, A \, ([I_{\mathbf{R}^2}]_\beta^\alpha)^{-1}.$$

The change of coordinates matrix $[I_{\mathbf{R}^2}]_\alpha^\beta$ has as columns the elements of β and so equals A. Therefore

$$[T]_\alpha = AAA^{-1} = A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}.$$

7. (10 points)

$P_1(\mathbf{R})$ is the real vector space of real polynomials of degree at most 1. Consider the linear maps $T : P_1(\mathbf{R}) \to P_1(\mathbf{R})$ and $S : P_1(\mathbf{R}) \to P_1(\mathbf{R})$ given by

$$T(p)(t) = 2p(t) + p'(t) \text{ and } S(p)(t) = p(t) + (t+1)p'(t).$$

Answer the following questions by performing calculations. You need not give a proof.

(a) Find the eigenvalues of T and S.

(b) Find the corresponding eigenvectors for T and S.

(c) Which of the linear transformations T, S are diagonalizable?

Answer:

The matrices representing T and S respectively in the standard basis $\beta = \{1, t\}$ are

$$[T]_\beta := A = \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix} \qquad [S]_\beta := B = \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix}.$$

We answer the questions for A and B and note what this implies about T and S.

(a) We begin by finding the eigenvalues of the two matrices.

Setting the characteristic polynomials to 0, we get (first for A)

$$0 = \det(A - tI) = \det\begin{pmatrix} 2-t & 1 \\ 0 & 2-t \end{pmatrix} = (2-t)^2$$

So A has only one eigenvalue, 2. The same holds for T.

For B, we get

$$0 = \det(B - tI) = \det\begin{pmatrix} 2-t & 1 \\ 0 & 1-t \end{pmatrix} = (2-t)(1-t)$$

so B has eigenvalues 2 and 1. The same holds for S.

(b) We begin by finding the eigenvectors of the two matrices.

First we deal with A, which has only one eigenvalue, $\lambda = 2$. If $v = \binom{x}{y}$ is an eigenvector for this eigenvalue, we must have $Av = 2v$, in other words

$$\begin{aligned} 2x + y &= 2x \\ 2y &= 2y. \end{aligned}$$

The second equation is always satisfied, so we need only consider the first equation. It simplifies to $y = 0$. x could be anything, so an eigenvalue could be $\binom{x}{0}$ for any nonzero value of x. Therefore, the non-zero constant polynomials of the form $p(t) = x$ are the only eigenvectors of T with eigenvalue 2.

Next consider B. For the eigenvalue $\lambda = 1$, if $v = \binom{x}{y}$ is an eigenvector then we get $Bv = 2v$, which gives

$$\begin{aligned} x + y &= x \\ 2y &= y \end{aligned}$$

The second equation gives $y = 0$ and then any value of x satisfies the first equation. Thus for any nonzero value of x, we get that $\binom{x}{0}$ is an eigenvector corresponding to $\lambda = 1$. Therefore, the non-zero constant polynomials of the form $p(t) = x$ are the only eigenvectors of S with eigenvalue 1.

For $\lambda = 2$, we get the equations

$$\begin{aligned} x + y &= 2x \\ 2y &= 2y \end{aligned}$$

The second equation is always true, and the first equation gives $x = y$. Therefore, for any nonzero value of x, we get that $\binom{x}{x}$ is an eigenvector corresponding to $\lambda = 2$. Therefore, the non-zero polynomials of the form $p(t) = x + xt$ are the only eigenvectors of S with eigenvalue 2.

(c) $\dim(P_1(\mathbf{R})) = 2$. T has only one eigenvector (up to scalar multiples), so T is not diagonalizable. S has two linearly independent eigenvectors, so S is diagonalizable.

8. (10 points)

Let A and B be real $n \times n$ square matrices.

(a) Suppose that AB is not invertible. Is it true that at least one of A and B is not invertible? Provide a proof or counter example.

(b) Suppose that A has at most $n-1$ nonzero entries, that is at most $n-1$ of the $A_{ij} \neq 0$. Is it true that $\det(A) = 0$? Provide a proof or counter example.

(c) Suppose that A and B commute, that is $AB = BA$. Is it true that $\det(A^2 - B^2) = \det(A - B)\det(A + B)$? Provide a proof or counter example.

(d) Suppose that $A^k = I_n$ for some positive integer $k > 0$. What are the possible values of $\det(A)$? Justify your answer.

Answer:

(a) True. AB is not invertible and so

$$0 = \det(AB) = \det(A)\det(B).$$

Therefore, at least one of $\det(A)$ and $\det(B)$ is zero. Consequently, at least one of A and B is not invertible.

(b) True. Use the definition of det.

$$\det(A) = \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) A_{1\sigma(1)} \ldots A_{n\sigma(n)}.$$

In each of the $n!$ products, there are n terms of the form $A_{i\sigma(i)}$. At least one of the $A_{i\sigma(i)}$ is zero and so each product is 0. Therefore, $\det(A) = 0$.

(c) True. The commutativity of A and B implies

$$(A - B)(A + B) = A^2 - AB + BA - B^2 = A^2 - B^2.$$

The multiplicativity of det implies the result:

$$\det(A^2 - B^2) = \det((A - B)(A + B)) = \det(A - B)\det(A + B).$$

(d) The possible values are $\det(A) = \pm 1$. Indeed,

$$1 = \det(I_n) = \det(A^k) = \det(A)^k.$$

$\det(A)$ is a real number, because all the entries of A are real numbers. Therefore, $\det(A)$ is a real root of unity and so $\det(A) = \pm 1$.

9. (10 points)

Recall that we say an $n \times n$ matrix A over the complex numbers is self-adjoint if $A^* = A$, where A^* is the complex conjugate of the transpose of A.

We call an $n \times n$ matrix A over the complex numbers a *Mueller-Petridis matrix* if $A^* = 3A$.

(a) Give an example of a 2×2 Mueller-Petridis matrix.

(b) Give a complete list of $n \times n$ Mueller-Petridis matrices, and prove your answer.

(c) Suppose A is an $n \times n$ matrix over the complex numbers, and $A^* = \lambda A$ for some scalar λ. What are the possible values of λ, and how does λ depend on the matrix A?

Answer:

(a) The only example is the 2×2 zero matrix.

(b) Since $(A^*)^* = A$, it follows that $A = (A^*)^* = (3A)^* = 3(A^*) = 9A$. Therefore, $9A_{ij} = A_{ij}$ for every entry A_{ij} of A. So A must be the $n \times n$ zero matrix.

(c) As in part (b), we have $A = (A^*)^* = (\lambda A)^* = \overline{\lambda}(A^*) = \overline{\lambda}\lambda A$. Recall that for a complex number λ, we have $\overline{\lambda}\lambda = |\lambda|^2$. So we can conclude that either $A = 0$, in which case all values of λ are allowed, or $A \neq 0$ and $|\lambda|^2 = 1$.

10. (10 points)

Consider the following basis for $\mathbf{R}^4$.

$$\beta = \left\{ \mathbf{w}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \mathbf{w}_2 = \begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \end{pmatrix}, \mathbf{w}_3 = \begin{pmatrix} 1 \\ -1 \\ 1 \\ 0 \end{pmatrix}, \mathbf{w}_4 = \begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix} \right\}.$$

(a) Apply the Gram-Schmidt orthonormalisation process to obtain an orthonormal basis $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$. Show all your work.

(b) Find an orthonormal basis for the orthogonal complement to

$$\operatorname{span}\left(\left\{\mathbf{w}_1 = \begin{pmatrix}1\\0\\0\\0\end{pmatrix}, \mathbf{w}_2 = \begin{pmatrix}1\\-1\\0\\0\end{pmatrix}\right\}\right).$$

Answer:

(a) The process gives

$$\{\mathbf{v}_1 = \mathbf{e}_1, \mathbf{v}_2 = -\mathbf{e}_2, \mathbf{v}_3 = \mathbf{e}_3, \mathbf{v}_4 = -\mathbf{e}_4\},$$

where $\mathbf{e}_i$ is the ith standard basis vector.

(b) By construction $\mathbf{v}_3$ and $\mathbf{v}_4$ are orthonormal vectors that lie in the orthogonal complement. They are 2 linearly independent vectors in a vector space of dimension $4 - 2 = 2$ and so $\{\mathbf{v}_3, \mathbf{v}_4\}$ is a basis.