# MTH 235: Linear Algebra

Final Exam

May 7, 2013

**NAME (please print legibly):** ______________________________

**Your University ID Number:** ______________________________

**Indicate your instructor with a check in the box:**

| | |
|---|---|
| Giorgis Petridis | |
| Mark Herman | |

- **There are no notes, textbooks, etc. allowed on this exam. The presence of calculators, cell phones, iPods and other electronic devices at this exam is strictly forbidden.**
- **Show your work and justify your answers. You may not receive full credit for a correct answer if insufficient work is shown or insufficient justification is given.**
- **Clearly circle or label your final answers.**

| QUESTION | VALUE | SCORE |
|---|---|---|
| 1 | 12 | |
| 2 | 10 | |
| 3 | 10 | |
| 4 | 12 | |
| 5 | 12 | |
| 6 | 10 | |
| 7 | 8 | |
| 8 | 12 | |
| 9 | 14 | |
| TOTAL | 100 | |

**1. (12 points) True or false?** Decide whether the following statements are true or false. If they are true prove them briefly (stating clearly any results you use); if they are false provide a counter example.

(a) Let $A$ and $B$ be $n \times n$ real matrices. Then, $\operatorname{rank}(AB) = \min\{\operatorname{rank}(A), \operatorname{rank}(B)\}$.

(b) If $A$ is an $n \times n$ real matrix, then $A$ and $A^t$ have the same eigenvalues.

(c) Let $V$ be an inner product space and $W \subset V$ be a subspace with basis $\{\mathbf{w}_1, \ldots, \mathbf{w}_k\}$ and $\mathbf{v} \in V$ be a vector that satisfies $\langle \mathbf{v}, \mathbf{w}_i \rangle = 0$ for all $i = 1, \ldots, k$. Then $\mathbf{v} \in W^{\perp}$.

**2. (10 points)** Let $V$ be a finite dimensional vector space and $W_1, W_2 \subset V$ be subspaces.

(a) Prove that $W_1 + W_2$ is a subspace of $V$.

(b) Suppose further that $W_1 + W_2 = V$. Can $\dim(W_1) + \dim(W_2) < \dim(V)$? Justify briefly.

**3. (10 points)** Let $V = M_{2\times 2}(\mathbb{R})$, the space of $2 \times 2$ real matrices, and

$$W = \{A \in M_{2\times 2}(\mathbb{R}) : \begin{pmatrix} 1 \\ 1 \end{pmatrix} \text{ is an eigenvector of } A\},$$

which is a subspace of $V$ (you do not need to prove this). Find a basis for $W$ showing your work. You do not need to justify why the set is a basis. Furthermore find the appropriate value of $n$ such that $W$ is isomorphic to $\mathbb{R}^n$.

**4. (12 points)** Let $T : V \to V$ be a linear operator on a vector space $V$.

(a) Show that if $R(T^k) = R(T^{k+1})$, then $R(T^{k+1}) = R(T^{k+2})$.

(b) Show that if $\dim(V) = 2$ and $T^2 \neq T_0$, where $T_0$ is the zero operator, then $T^k \neq T_0$ for all $k = 1, 2, \ldots$

**5. (12 points)** Let $V, W$ be finite dimensional vector spaces over the same field, $\beta = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ be a basis for $V$ and $T : V \to W$ be a linear one-to-one map.

(a) Show that $T(\beta) = \{T(\mathbf{v}_1), \ldots, T(\mathbf{v}_n)\}$ is a linearly independent set in $W$.

(b) Assuming further that $\dim(V) = \dim(W)$, prove that $T(\beta)$ is a basis for $W$.

(c) Use part (b) to construct bases $\alpha$ and $\gamma$ for $V$ and $W$ respectively such that $[T]_\alpha^\gamma$ is the identity matrix.

**6. (10 points)** You are given a vector space $V$ with an ordered basis $\beta$ and a linear operator $T : V \to V$. Find the matrix $[T]_\beta$ showing your work.

(a) $V = \mathbb{R}^3$, $\beta = \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right\}$ and $T \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} a+b \\ b+c \\ c+a \end{pmatrix}$.

(b) $V = M_{2\times 2}(\mathbb{R})$, the space of $2\times 2$ real matrices, $\beta = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}$ and $T(A) = A^t + 2A$.

**7. (8 points)** Find $k$ that satisfies:

$$\det \begin{pmatrix} b_3 - c_3 & b_2 - c_2 & b_1 - c_1 \\ a_3 + 3c_3 & a_2 + 3c_2 & a_1 + 3c_1 \\ a_3 + b_3 + c_3 & a_2 + b_2 + c_2 & a_1 + b_1 + c_1 \end{pmatrix} = k \det \begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{pmatrix}.$$

Show your work.

**8. (12 points)** Let $W = \operatorname{span}\left\{ \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\}$ and endow it with the (usual) dot product.

(a) Find an orthonormal basis for $W$.

(b) Let $P_W$ be the projection on $W$. Calculate $P_W\left(\begin{pmatrix} 4 \\ 0 \\ 0 \\ 0 \end{pmatrix}\right)$.

**9. (14 points)** In this question you are asked to diagonalize a linear map and a symmetric matrix.

(a) Let $T : P_2 \to P_2$ be the linear operator given by $T(p)(x) = p(0) + p'(1)x + p''(2)x^2$, where $P_2$ is the space of real polynomials of degree at most 2. Find a basis $\beta$ such that $[T]_\beta$ is diagonal and also write down the diagonal matrix $[T]_\beta$.

(b) Let $A = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ -1 & 1 & 0 \end{pmatrix}$, with characteristic polynomial $(1+t)(1-t)(2-t)$. Find a matrix $P$ such that $D = P^t AP$ is a diagonal matrix and also write down $D$.