MTH 235: Linear Algebra

Final Exam

May 7, 2013

NAME (please print legibly): Solutions

Your University ID Number:

Indicate your instructor with a check in the box:

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- There are no notes, textbooks, etc. allowed on this exam. The presence of calculators, cell phones, iPods and other electronic devices at this exam is strictly forbidden.
- Show your work and justify your answers. You may not receive full credit for a correct answer if insufficient work is shown or insufficient justification is given.
- Clearly circle or label your final answers.

QUESTION	VALUE	SCORE
1	12	
2	10	
3	10	
4	12	
5	12	
6	10	
7	8	
8	12	
9	14	
TOTAL	100	

1. (12 points) True or false? Decide whether the following statements are true or false. If they are true prove them briefly (stating clearly any results you use); if they are false provide a counter example.

(a) Let A and B be $n \times n$ real matrices. Then, $\text{rank}(AB) = \min\{\text{rank}(A), \text{rank}(B)\}$.

False. $A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ $B = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$

$\text{rank}(A) = \text{rank}(B) = 1$

Yet $AB = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ of rank zero.

(b) If A is an $n \times n$ real matrix, then A and A^t have the same eigenvalues.

True

$\chi_A(t) = \det(A - tI) = \det((A - tI)^t)$

$= \det(A^t - tI) = \chi_{A^t}(t)$

So A, A^t have the same characteristic polynomial and so have the same eigenvalues.

(c) Let V be an inner product space and $W \subset V$ be a subspace with basis $\{\mathbf{w}_1, \ldots, \mathbf{w}_k\}$ and $\mathbf{v} \in V$ be a vector that satisfies $\langle \mathbf{v}, \mathbf{w}_i \rangle = 0$ for all $i = 1, \ldots, k$. Then $\mathbf{v} \in W^\perp$.

True

Let $\vec{u} \in W$. $\vec{u} = \sum_{i=1}^{k} a_i \vec{w_i}$

$\langle \vec{v}, \vec{u} \rangle = \langle \vec{v}, \sum_{i=1}^{k} a_i \vec{w_i} \rangle = \sum_{i=1}^{k} a_i \langle \vec{v}, \vec{w_i} \rangle$

$= \sum_{i=1}^{k} a_i \cdot 0 = 0$.

2. (10 points) Let V be a finite dimensional vector space and $W_1, W_2 \subset V$ be subspaces.

(a) Prove that $W_1 + W_2$ is a subspace of V.

- $\vec{0} \in W_1 + W_2$ because $\underset{\text{in } W_1}{\vec{0}} = \vec{0} + \underset{\text{in } W_2}{\vec{0}}$

- Let $\vec{x} = \vec{w_1} + \vec{w_2}$, $\vec{y} = \vec{w_1}' + \vec{w_2}'$ be elements of $W_1 + W_2$. Their sum $\vec{x} + \vec{y} \in W_1 + W_2$ because

$$\vec{x} + \vec{y} = \underbrace{(\vec{w_1} + \vec{w_1}')}_{\text{in } W_1} + \underbrace{(\vec{w_2} + \vec{w_2}')}_{\text{in } W_2} \in W_1 + W_2$$

because W_1 is a subspace ... because W_2 is a subspace

- Similarly if $\vec{x} = \vec{w_1} + \vec{w_2}$, $\lambda \in F$, then $\lambda\vec{x} = \lambda\vec{w_1} + \lambda\vec{w_2} \in W_1 + W_2$ because W_1, W_2 are subspaces.

(b) Suppose further that $W_1 + W_2 = V$. Can $\dim(W_1) + \dim(W_2) < \dim(V)$? Justify briefly.

No. Let β be a basis for W_1
γ — — — — W_2.

$W_1 + W_2 = V$ implies that $\beta \cup \gamma$ spans V.

So $\dim(V) \leq |\beta \cup \gamma| \leq |\beta| + |\gamma| = \dim(W_1) + \dim(W_2)$.

3. (10 points) Let $V = M_{2\times 2}(\mathbb{R})$, the space of 2×2 real matrices, and

$$W = \{A \in M_{2\times 2}(\mathbb{R}) : \begin{pmatrix} 1 \\ 1 \end{pmatrix} \text{ is an eigenvector of } A\},$$

which is a subspace of V (you do not need to prove this). Find a basis for W showing your work. You do not need to justify why the set is a basis. Furthermore find the appropriate value of n such that W is isomorphic to $\mathbb{R}^n$.

Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in W$. Then

$A \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \lambda \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ implies $\begin{cases} a+b=\lambda \\ c+d=\lambda \end{cases}$ which is equivalent

to $a+b=c+d$ or $d=a+b-c$. So

$A = \begin{pmatrix} a & b \\ c & a+b-c \end{pmatrix} \in \text{span}\left\{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & -1 \end{pmatrix}\right\}$.

The set is linearly independent and so is a basis

$\dim(W) = 3 \iff W$ isomorphic to $\mathbb{R}^3$.

4. (12 points) Let $T : V \to V$ be a linear operator on a vector space V.

(a) Show that if $R(T^k) = R(T^{k+1})$, then $R(T^{k+1}) = R(T^{k+2})$.

$$R(T^{k+2}) = T[R(T^{k+1})]$$
$$= T[R(T^k)]$$
$$= R(T^{k+1})$$

(b) Show that if $\dim(V) = 2$ and $T^2 \neq T_0$, where T_0 is the zero operator, then $T^k \neq T_0$ for all $k = 1, 2, \ldots$

$T \neq T_0$ as $T = T_0$ implies $T^2 = T_0$.

Therefore, rank $(T) = 1$ or 2.

- If rank $(T) = 2$, then T is an isomorphism and so are T^2, T^3, etc. So $T^k \neq T_0$ for all k.
- If rank $(T) = 1$, we show $R(T) = R(T^2)$

$R(T) \supseteq R(T^2)$ is always true.

$T^2 \neq T_0$ and so nullity$(T^2) \leq 1$.

By Rank-Nullity rank$(T^2) = 2 -$ nullity$(T^2) \geq 1$.

So rank $(T) =$ rank (T^2) which implies $T = T^2$

By part (a) $R(T^k) = R(T) \neq \{\vec{0}\}$ so $T_k \neq T_0$.

5. (12 points) Let V, W be finite dimensional vector spaces over the same field, $\beta = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ be a basis for V and $T : V \to W$ be a linear one-to-one map.

(a) Show that $T(\beta) = \{T(\mathbf{v}_1), \ldots, T(\mathbf{v}_n)\}$ is a linearly independent set in W.

Suppose $\vec{0} = \sum_{i=1}^{n} \lambda_i T(\vec{v_i})$ We show $\lambda_1 = \cdots = \lambda_n = 0$.

$$T(\vec{0}) = \vec{0} = \sum_{i=1}^{n} \lambda_i T(\vec{v_i}) = T\left(\sum_{i=1}^{n} \lambda_i \vec{v_i}\right).$$

T is one-to-one and so

$$\vec{0} = \sum_{i=1}^{n} \lambda_i \vec{v_i}.$$

β is linearly independent and so $\lambda_1 = \cdots = \lambda_n = 0$.

(b) Assuming further that $\dim(V) = \dim(W)$, prove that $T(\beta)$ is a basis for W.

The elements of $T(\beta)$ are distinct.

There are $n = \dim(V) = \dim(W)$ of them

and so they form a basis for W.

(c) Use part (b) to construct bases α and γ for V and W respectively such that $[T]_\alpha^\gamma$ is the identity matrix.

Let $\alpha = \beta$

$\gamma = T(\beta)$. Then

$$[T]_\alpha^\beta = \left([T(\vec{v_1})]_\gamma \Big| \cdots \Big| [T(\vec{v_n})]_\gamma \right)$$

$$= \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & & 0 \\ 0 & 0 & & \vdots \\ \vdots & \vdots & & 0 \\ 0 & 0 & & 1 \end{pmatrix}$$

$$= I_n$$

6. (10 points) You are given a vector space V with an ordered basis β and a linear operator $T : V \to V$. Find the matrix $[T]_\beta$ showing your work.

(a) $V = \mathbb{R}^3$, $\beta = \left\{ \begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}1\\1\\0\end{pmatrix}, \begin{pmatrix}1\\1\\1\end{pmatrix} \right\}$ and $T\begin{pmatrix}a\\b\\c\end{pmatrix} = \begin{pmatrix}a+b\\b+c\\c+a\end{pmatrix}$.

$T\begin{pmatrix}1\\0\\0\end{pmatrix} = \begin{pmatrix}1\\0\\1\end{pmatrix} = \begin{pmatrix}1\\0\\0\end{pmatrix} - \begin{pmatrix}1\\1\\0\end{pmatrix} + \begin{pmatrix}1\\1\\1\end{pmatrix}$ so $\left[T\begin{pmatrix}1\\0\\0\end{pmatrix}\right]_\beta = \begin{pmatrix}1\\-1\\1\end{pmatrix}$

$T\begin{pmatrix}1\\1\\0\end{pmatrix} = \begin{pmatrix}2\\1\\1\end{pmatrix} = \begin{pmatrix}1\\0\\0\end{pmatrix} + \begin{pmatrix}1\\1\\1\end{pmatrix}$ so $\left[T\begin{pmatrix}1\\1\\0\end{pmatrix}\right]_\beta = \begin{pmatrix}1\\0\\1\end{pmatrix}$

$T\begin{pmatrix}1\\1\\1\end{pmatrix} = \begin{pmatrix}2\\2\\2\end{pmatrix} = 2\begin{pmatrix}1\\1\\1\end{pmatrix}$ so $\left[T\begin{pmatrix}1\\1\\1\end{pmatrix}\right]_\beta = \begin{pmatrix}0\\0\\2\end{pmatrix}$

Therefore $[T]_\beta = \begin{pmatrix}1 & 1 & 0\\ -1 & 0 & 0\\ 1 & 1 & 2\end{pmatrix}$

(b) $V = M_{2\times2}(\mathbb{R})$, the space of 2×2 real matrices, $\beta = \left\{ \begin{pmatrix}1&0\\0&0\end{pmatrix}, \begin{pmatrix}0&1\\0&0\end{pmatrix}, \begin{pmatrix}0&0\\1&0\end{pmatrix}, \begin{pmatrix}0&0\\0&1\end{pmatrix} \right\}$ and $T(A) = A^t + 2A$.

$T\begin{pmatrix}1&0\\0&0\end{pmatrix} = \begin{pmatrix}3&0\\0&0\end{pmatrix}$ so $\left[T\begin{pmatrix}1&0\\0&0\end{pmatrix}\right]_\beta = \begin{pmatrix}3\\0\\0\\0\end{pmatrix}$

$T\begin{pmatrix}0&1\\0&0\end{pmatrix} = \begin{pmatrix}0&2\\1&0\end{pmatrix}$ so $\left[T\begin{pmatrix}0&1\\0&0\end{pmatrix}\right]_\beta = \begin{pmatrix}0\\2\\1\\0\end{pmatrix}$

$T\begin{pmatrix}0&0\\1&0\end{pmatrix} = \begin{pmatrix}0&1\\2&0\end{pmatrix}$ so $\left[T\begin{pmatrix}0&0\\1&0\end{pmatrix}\right]_\beta = \begin{pmatrix}0\\1\\2\\0\end{pmatrix}$

$T\begin{pmatrix}0&0\\0&1\end{pmatrix} = \begin{pmatrix}0&0\\0&3\end{pmatrix}$ so $\left[T\begin{pmatrix}0&0\\0&1\end{pmatrix}\right]_\beta = \begin{pmatrix}0\\0\\0\\3\end{pmatrix}$

Therefore $[T]_\beta = \begin{pmatrix}3&0&0&0\\0&2&1&0\\0&1&2&0\\0&0&0&3\end{pmatrix}$

7. (8 points) Find k that satisfies:

$$\det\begin{pmatrix} b_3-c_3 & b_2-c_2 & b_1-c_1 \\ a_3+3c_3 & a_2+3c_2 & a_1+3c_1 \\ a_3+b_3+c_3 & a_2+b_2+c_2 & a_1+b_1+c_1 \end{pmatrix} = k\det\begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{pmatrix}.$$

Show your work.

$$\Delta = \begin{vmatrix} b_3-c_3 & b_2-c_2 & b_1-c_1 \\ a_3+3c_3 & a_2+3c_2 & a_1+3c_1 \\ a_3+b_3+c_3 & a_2+b_2+c_2 & a_1+b_1+c_1 \end{vmatrix} \overset{C_1\leftrightarrow C_3}{=} - \begin{vmatrix} b_1-c_1 & b_2-c_2 & b_3-c_3 \\ a_1+3c_1 & a_2+3c_2 & a_3+3c_3 \\ a_1+b_1+c_1 & a_2+b_2+c_2 & a_3+b_3+c_3 \end{vmatrix}$$

Using row notation

$$\Delta = -\begin{vmatrix} \vec{b}-\vec{c} \\ \vec{a}+3\vec{c} \\ \vec{a}+\vec{b}+\vec{c} \end{vmatrix} \overset{R_1\leftrightarrow R_2}{=} \begin{vmatrix} \vec{a}+3\vec{c} \\ \vec{b}-\vec{c} \\ \vec{a}+\vec{b}+\vec{c} \end{vmatrix} \overset{R_3=r_3-r_1}{=} \begin{vmatrix} \vec{a}+3\vec{c} \\ \vec{b}-\vec{c} \\ \vec{b}-2\vec{c} \end{vmatrix}$$

$$\overset{R_3=r_3-r_2}{=} \begin{vmatrix} \vec{a}+3\vec{c} \\ \vec{b}-\vec{c} \\ -\vec{c} \end{vmatrix} = -\begin{vmatrix} \vec{a}+3\vec{c} \\ \vec{b}-\vec{c} \\ \vec{c} \end{vmatrix} \overset{R_2=r_2+r_3}{=} -\begin{vmatrix} \vec{a}+3\vec{c} \\ \vec{b} \\ \vec{c} \end{vmatrix}$$

$$\overset{R_1=r_1-3r_3}{=} -\begin{vmatrix} \vec{a} \\ \vec{b} \\ \vec{c} \end{vmatrix}$$

So $k=-1$.

8. (12 points) Let $W = \text{span}\left\{ \begin{pmatrix}1\\0\\1\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\\1\end{pmatrix}, \begin{pmatrix}1\\0\\0\\1\end{pmatrix} \right\}$ and endow it with the (usual) dot product.

$\vec{w_1} \quad \vec{w_2} \quad \vec{w_3}$

(a) Find an orthonormal basis for W.

Apply Gram-Schmidt.

$$\vec{u_1} = \frac{\vec{w_1}}{\|\vec{w_1}\|} = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\0\\1\\0\end{pmatrix}$$

$$\vec{v_2} = \vec{w_2} - \langle \vec{w_2}, \vec{u_1} \rangle \cdot \vec{u_1} = \begin{pmatrix}0\\1\\0\\1\end{pmatrix} - 0 \cdot \begin{pmatrix}1\\0\\1\\0\end{pmatrix} = \begin{pmatrix}0\\1\\0\\1\end{pmatrix}$$

$$\vec{u_2} = \frac{\vec{v_2}}{\|\vec{v_2}\|} = \frac{1}{\sqrt{2}}\begin{pmatrix}0\\1\\0\\1\end{pmatrix}$$

$$\vec{v_3} = \vec{w_3} - \langle \vec{w_3}, \vec{u_2} \rangle \vec{u_2} - \langle \vec{w_3}, \vec{u_1} \rangle \vec{u_1}$$

$$= \begin{pmatrix}1\\0\\0\\1\end{pmatrix} - \frac{1}{2}\begin{pmatrix}0\\1\\0\\1\end{pmatrix} - \frac{1}{2}\begin{pmatrix}1\\0\\1\\0\end{pmatrix} = \frac{1}{2}\begin{pmatrix}1\\-1\\-1\\1\end{pmatrix}$$

$\vec{u_3} = \vec{v_3}$.

An orthonormal basis is $\left\{ \frac{1}{\sqrt{2}}\begin{pmatrix}1\\0\\1\\0\end{pmatrix}, \frac{1}{\sqrt{2}}\begin{pmatrix}0\\1\\0\\1\end{pmatrix}, \frac{1}{2}\begin{pmatrix}1\\-1\\-1\\1\end{pmatrix} \right\}$

(b) Let P_W be the projection on W. Calculate $P_W\left(\underbrace{\begin{pmatrix}4\\0\\0\\0\end{pmatrix}}_{=\vec{y}}\right)$.

$$P_W\left(\begin{pmatrix}4\\0\\0\\0\end{pmatrix}\right) = P_W(\vec{y}) = \sum_{i=1}^{3}\langle \vec{y}, \vec{u_i}\rangle \vec{u_i}$$

$$= \begin{pmatrix}2\\0\\2\\0\end{pmatrix} + \begin{pmatrix}0\\0\\0\\0\end{pmatrix} + \begin{pmatrix}1\\-1\\-1\\1\end{pmatrix}$$

$$= \begin{pmatrix}3\\-1\\1\\1\end{pmatrix}$$

Note: $\vec{y} - P_W(\vec{y}) = \begin{pmatrix}1\\1\\-1\\-1\end{pmatrix}$ is orthogonal to W as it ought to be.

9. (14 points) In this question you are asked to diagonalize a linear map and a symmetric matrix.

(a) Let $T : P_2 \to P_2$ be the linear operator given by $T(p)(x) = p(0) + p'(1)x + p''(2)x^2$, where P_2 is the space of real polynomials of degree at most 2. Find a basis β such that $[T]_\beta$ is diagonal and also write down the diagonal matrix $[T]_\beta$.

$\alpha = \{1, x, x^2\}$

$$[T]_\alpha = \left([T(1)]_\alpha \,\middle|\, [T(x)]_\alpha \,\middle|\, [T(x^2)]_\alpha \right)$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 2 \end{pmatrix}$$

The eigenvalues of T are the eigenvalues of $[T]_\alpha$. $[T]_\alpha$ is upper triangular and so its eigenvalues are its diagonal elements $\lambda = 1, 2$.

$[T]_\alpha$ $\lambda = 1$ eigenvectors: $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ by inspection.

$[T]_\alpha$ $\lambda = 2$ eigenvector: $[T]_\alpha - 2I = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 2 \\ 0 & 0 & 0 \end{pmatrix}$

the eigenvector is $\begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix}$.

The eigenvectors of T are $\{1, x, 2x + x^2\}$

So we take $\boxed{\beta = \{1, x, 2x + x^2\}}$ and

$$\boxed{[T]_\beta = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}}$$

(b) Let $A = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ -1 & 1 & 0 \end{pmatrix}$, with characteristic polynomial $(1+t)(1-t)(2-t)$. Find a matrix P such that $D = P^tAP$ is a diagonal matrix and also write down D.

Eigenvectors of A

$\lambda = -1$: $A + I = \begin{pmatrix} 2 & 0 & -1 \\ 0 & 2 & 1 \\ -1 & 1 & 1 \end{pmatrix}$ eigenvector is $\begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$

$\lambda = 1$: $A - I = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 1 \\ -1 & 1 & -1 \end{pmatrix}$ ... $\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$

$\lambda = 2$: $A - 2I = \begin{pmatrix} -1 & 0 & -1 \\ 0 & -1 & 1 \\ -1 & 1 & -2 \end{pmatrix}$ ... $\begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix}$

P has as columns the normalized eigenvectors.

So $P = \begin{pmatrix} \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{3}} \\ 2/\sqrt{6} & 0 & -\frac{1}{\sqrt{3}} \end{pmatrix}$

$D = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$