Math 235: Midterm 2: Solutions

University of Rochester

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Question:	1	2	3	4	5	Total
Points:	10	20	20	30	20	100

1. (10 points) Select either true or false, completely filling in the relevant bubble.

   (a) An $m \times n$ system $Ax = 0$ is always consistent.

   ◯ True ◯ False $\surd$ **True**

   Solution:
   The zero vector in $\mathbb{F}^n$ is always a solution to $Ax = 0$.

   (b) Let V be a finite dimensional vector space and $T\colon V \to V$ be a linear transformation. Then, $\det[T]_\beta = \det[T]_\gamma$ for any two ordered bases β and γ of V.

   ◯ True ◯ False $\surd$ **True**

   Solution:
   Any two matrix representations of a linear transformation are similar, and thus have the same determinant.

   (c) If λ is an eigenvalue of a linear operator T, then λ^n is an eigenvalue of T^n for all $n \geq 1$.

   ◯ True ◯ False $\surd$ **True**

   Solution:
   Proof by induction on n. We show that if v is an eigenvector of T with eigenvalue λ, then v is an eigenvector of T^n with eigenvalue λ^n. If $n = 1$ the result is trivial. Suppose $n > 1$. By the inductive hypothesis, v is an eigenvector of T^{n-1} with eigenvalue λ^{n-1}. Then, $T^n(v) = T(T^{n-1}(v)) = T(\lambda^{n-1}v) = \lambda^{n-1}T(v) = \lambda^{n-1}\lambda v = \lambda^n v$.a

   (d) Let A be a square matrix with reduced row echelon form B. Then, $\det A = \det B$.

   ◯ True ◯ False $\surd$ **False**

Solution:
Not all elementary row operations preserve determinants.

(e) Let $T\colon V \to W$ be a linear transformation. If $\dim(V) = \dim(W)$, then T is invertible.

◯ True ◯ False $\surd$ **False**

Solution:
Let T be the zero transformation from V to W. This is not an isomorphism if $\dim V, \dim W > 0$.

2. Let $B = \begin{bmatrix} -1 & 1 \\ 0 & 1 \end{bmatrix} \in M_2(\mathbb{R})$. Define the linear transformation $T_B\colon M_2(\mathbb{R}) \to \mathbb{R}$ by

$$T_B(A) = \operatorname{tr}(AB).$$

(a) (5 points) Compute $\operatorname{rank}(T_B)$.

Solution:
Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in M_2(\mathbb{R})$. Then,

$$T_B(A) = \operatorname{tr}\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} -1 & 1 \\ 0 & 1 \end{bmatrix}\right) = \operatorname{tr}\begin{bmatrix} -a & a+b \\ -c & c+d \end{bmatrix} = -a + c + d.$$

In particular, we can let $a = c = 0$, and letting d range through all real numbers we see that T_B is onto, so $\operatorname{rank}(T_B) = 1$.

(b) (5 points) Show that $A = \begin{bmatrix} 1 & -1 \\ 2 & -1 \end{bmatrix}$ is in the nullspace $N(T_B)$.

Solution:
This is a pure calculation. If A is as given in (b), then the equation in the solution of (a) shows that $T_B(A) = -1 + 2 - 1 = 0$.

(c) (10 points) Find a basis for $N(T_B)$ which contains A.

Solution:
By the Dimension Theorem, $\dim N(T_B) = 3$. Let $A' = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$ and $A'' = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. By the calculation in (a), both of these are in $N(T_B)$. Considering the top right entries of the matrices A, A' and A'', we see that any linear combination of those three to the zero matrix would have a coefficient of zero for A. The matrices

A' and A'' are clearly linearly independent, so the remaining coefficients of A' and A'' would also have to be zero. Hence $\{A, A', A''\}$ is a set of three linearly independent matrices in the three dimensional space $N(T_B)$, and is thus a basis for $N(T_B)$.

3. Let $A \in M_{3\times 4}(\mathbb{C})$ be the matrix whose reduced row echelon form is given by

$$\begin{bmatrix} 1 & i & 0 & 1 \\ 0 & 0 & 1 & -i \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Assume that the first and third columns, respectively, of A are

$$\begin{bmatrix} i \\ -1 \\ i \end{bmatrix} \text{ and } \begin{bmatrix} 1+i \\ 1-i \\ 2i \end{bmatrix}.$$

(a) (10 points) Find the matrix A.

Solution:
The second column of A is

$$i\begin{bmatrix} i \\ -1 \\ i \end{bmatrix} = \begin{bmatrix} -1 \\ -i \\ -1 \end{bmatrix}$$

while the fourth column of A is

$$\begin{bmatrix} i \\ -1 \\ i \end{bmatrix} - i\begin{bmatrix} 1+i \\ 1-i \\ 2i \end{bmatrix} = \begin{bmatrix} i-(i-1) \\ -1-(i+1) \\ i-(-2) \end{bmatrix} = \begin{bmatrix} 1 \\ -2-i \\ 2+i \end{bmatrix}.$$

Hence,

$$A = \begin{bmatrix} i & -1 & 1+i & 1 \\ -1 & -i & 1-i & -2-i \\ i & -1 & 2i & 2+i \end{bmatrix}$$

(b) (10 points) Find a basis for the solution space to the system $Ax = 0$.

Solution:
From the reduced row echelon form of A, the solution space for $Ax = 0$ is

$$\left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \in \mathbb{C}^4 : x_1 = -ix_2 - x_4, x_2 = z, x_3 = ix_4, x_4 = w \text{ with } z, w \in \mathbb{C} \right\}$$

In particular, any such solution is of the form

$$\begin{bmatrix} -iz - w \\ z \\ iw \\ w \end{bmatrix} = z \begin{bmatrix} -i \\ 1 \\ 0 \\ 0 \end{bmatrix} + w \begin{bmatrix} -1 \\ 0 \\ i \\ 1 \end{bmatrix}$$

Hence, the vectors $(-1, 0, i, 1)$ and $(-i, 1, 0, 0)$ span the solution space to $Ax = 0$, and they are evidently linearly independent, so it follows that they are a basis for the solution space.

4. Let

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 0 \\ 1 & 0 & 3 \end{bmatrix} \in M_3(\mathbb{R}).$$

(a) (10 points) Compute the characteristic polynomial $f(t)$ of A.

Solution:
We have $f(t) = \det(A - tI)$, so

$$f(t) = \det \begin{bmatrix} 1-t & 1 & 0 \\ 0 & 2-t & 0 \\ 1 & 0 & 3-t \end{bmatrix}.$$

Expanding along the third column,

$$f(t) = (3-t) \det \begin{bmatrix} 1-t & 1 \\ 0 & 2-t \end{bmatrix} = (3-t)(1-t)(2-t)$$

(b) (10 points) Find the eigenvalues of A.

Solution:
The roots of $f(t)$ are $1, 2$ and 3, so those are the eigenvalues of A.

(c) (10 points) Determine if A is diagonalizable. Justify your answer.

Solution:
The matrix A is diagonalizable, since it is a 3×3 matrix with three distinct eigenvalues.

5. (a) (10 points) Let V and W be vector spaces over a field $\mathbb{F}$, and let $T, U : V \to W$ be linear transformations. Show that $\operatorname{rank}(T + U) \leq \operatorname{rank}(T) + \operatorname{rank}(U)$.

Solution:
Let $v \in V$. Then, $(T+U)(v) = T(v) + U(v)$, so $R(T+U) \subseteq R(T) + R(U)$, so $\operatorname{rank}(T+U) = \dim R(T+U) \leq \dim(R(T)+R(U))$. Since W is finite dimensional, the dimension of a sum of subspaces is less than or equal to the sum of their dimensions, so $\dim(R(T)+R(U)) \leq \dim R(T) + \dim R(U) = \operatorname{rank}(T) + \operatorname{rank}(U)$. It follows that $\operatorname{rank}(T+U) \leq \operatorname{rank}(T) + \operatorname{rank}(U)$.

(b) (10 points) Let $A, B \in M_{m\times n}(\mathbb{F})$. Show that $\operatorname{rank}(A+B) \leq \operatorname{rank}(A) + \operatorname{rank}(B)$.

Solution:
Let $L_A, L_B \colon \mathbb{F}^n \to \mathbb{F}^m$ be the linear transformations associated to the matrices A and B. By definition, $\operatorname{rank}(A) = \operatorname{rank}(L_A), \operatorname{rank}(B) = \operatorname{rank}(L_B)$, and $\operatorname{rank}(A+B) = \operatorname{rank}(L_{A+B}) = \operatorname{rank}(L_A + L_B)$. By (a), $\operatorname{rank}(L_A + L_B) \leq \operatorname{rank}(L_A) + \operatorname{rank}(L_B)$, so $\operatorname{rank}(A+B) \leq \operatorname{rank}(A) + \operatorname{rank}(B)$.