Math 235: Midterm 1

University of Rochester

October 5, 2021

Name: ______________________________

UR ID: ______________________________

- You are allowed one page, single-sided of notes. No other resources are permitted.
- The exam questions are on pages 2-11 of this packet.
- Each part of questions 2 through 6 is on its own page. All work you want graded for that problem should be contained entirely on that page, unless:
- If you need more space on a problem, use the **Scratch work** pages at the end of the exam, and make sure to make a note on the problem page that you are doing so.
- **Do not tear off the scratch work pages.**
- Copy and sign the Honor Pledge: *I affirm that I will not give or receive any unauthorized help on this exam, and that all work will be my own.*

____

____

____

Signature: ______________________________

Question:	1	2	3	4	5	6	Total
Points:	10	20	15	15	10	30	100

1. (10 points) Select either true or false, completely filling in the relevant bubble.

   (a) The dimension of $\mathbb{C}$ (the set of complex numbers) as vector space over $\mathbb{R}$ (the set of real numbers) is 2.

   ◯ True ◯ False

   (b) Let $V = P(\mathbb{R})$. If $p(x) \in V$, then the set $\{p(x), xp(x)\}$ is linearly dependent in V.

   ◯ True ◯ False

   (c) The vector $x^3 - x$ belongs to $\mathrm{Span}(x^3 + x^2 + x, x^2 + 2x, x^2)$ in $P_3(\mathbb{F})$, for any field $\mathbb{F}$.

   ◯ True ◯ False

   (d) If V is a finite dimensional vector space of dimension n, then V contains a subspace of each dimension i for each $i = 0, \ldots, n$.

   ◯ True ◯ False

   (e) If V is a finite dimensional vector space over $\mathbb{F}_2$ and $\dim(V) = n$, then V contains precisely 2^n vectors.

   ◯ True ◯ False

2. (a) (10 points) Let V be a finite dimensional vector space over a field $\mathbb{F}$ and let W be a subspace of V. Show that if $\dim(W) = \dim(V)$, then $W = V$.

(b) (10 points) Let V be a finite dimensional vector space over a field $\mathbb{F}$ and let W_1 and W_2 be subspaces of V. Show that if $\dim(W_1 \cap W_2) = \dim(W_1 + W_2)$, then $W_1 = W_2$.

3. (15 points) Let V be a vector space over a field $\mathbb{F}$, and let w, x, y and z be vectors in V. Prove that the set $\{w + x, x + y, y + z, z + w\}$ is linearly **dependent** in V.

4. (15 points) Let $A \in M_{m\times n}(\mathbb{R})$ be a matrix and $\lambda \in \mathbb{R}$ be a scalar. Show that the set $W = \{v \in \mathbb{R}^n \mid Av = \lambda v\}$ is a subspace of $\mathbb{R}^n$.

5. (10 points) Let V and W be vector spaces over a field $\mathbb{F}$. Show that a linear transformation $T\colon V \to W$ is one-to-one if and only if T carries linearly independent subsets of V onto linearly independent subsets of W.

6. Let $\beta = \{x^2 - x, x - 1, 2x\}$ be a set of vectors in $P_2(\mathbb{R})$.

   (a) (10 points) Show that β is a basis for $P_2(\mathbb{R})$.

(b) (5 points) Let γ be the standard ordered basis of $\mathbb{R}^2$ and $\beta = \{x^2 - x, x - 1, 2x\}$ be the basis for $P_2(\mathbb{R})$ given in (a). If $T\colon P_2(\mathbb{R}) \to \mathbb{R}^2$ is a linear transformation such that $[T]_\beta^\gamma = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -2 \end{bmatrix}$, find $T(x^2 + x + 1)$.

(c) (10 points) Find a basis for $N(T)$ and determine if T is one-to-one.

(d) (5 points) Is T onto? Find $\dim(R(T))$.

Scratch work (first page)

Scratch work (second page) — DO NOT REMOVE

Scratch work (third page) — DO NOT REMOVE