Math 235: Final Exam

University of Rochester

December 12, 2021

Name: ______________________________

UR ID: ______________________________

- You are allowed one page, single-sided of notes. No other resources are permitted.
- The exam questions are on pages 2-16 of this packet.
- Each part of each question is on its own page. All work you want graded for that problem should be contained entirely on that page, unless:
- If you need more space on a problem, use the **Scratch work** pages at the end of the exam, and make sure to make a note on the problem page that you are doing so.
- **Do not tear off the scratch work pages.**
- Copy and sign the Honor Pledge: *I affirm that I will not give or receive any unauthorized help on this exam, and that all work will be my own.*

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Signature: ______________________________

Question:	1	2	3	4	5	6	Total
Points:	20	15	10	20	15	20	100

1. (20 points) Select either true or false, completely filling in the relevant bubble.

   (a) The function $T\colon (\mathbb{F}_2)^2 \to (\mathbb{F}_2)^2$ given by $T(x, y) = (x + y, x^2 + y^2)$ is a linear transformation.

   ◯ True ◯ False

   (b) If $T\colon V \to V$ is a linear transformation such that $T^k = I$ for some integer $k \geq 2$, then T is invertible.

   ◯ True ◯ False

   (c) Let $W \subseteq V$ be a subspace such that $\dim W = 0$. Then $\beta = \{0_V\}$ is a basis for W.

   ◯ True ◯ False

   (d) Any linear transformation $T\colon V \to W$ has a matrix representation in some space $M_n(\mathbb{F})$.

   ◯ True ◯ False

   (e) The set of solutions of a linear system $Ax = b$ is always a subspace.

   ◯ True ◯ False

   (f) If the system $m \times n$ system $Ax = b$ is consistent, then it has a unique solution.

   ◯ True ◯ False

(g) The function $\det \colon M_n(\mathbb{F}) \to \mathbb{F}$ is a linear transformation.

○ True ○ False

(h) If $A \in M_n(\mathbb{R})$, then $\det(AA^T) \geq 0$.

○ True ○ False

(i) If A and B are similar matrices in $M_n(\mathbb{F})$, then A and B have the same eigenvalues.

○ True ○ False

(j) If u and v be eigenvectors of a matrix $A \in M_n(\mathbb{R})$ associated to distinct eigenvalues λ and μ, then u and v are orthogonal.

○ True ○ False

(k) Let $u, v, w \in V$ be vectors in an inner product space. If $u \perp v$ and $v \perp w$, then $u \perp w$.

○ True ○ False

(l) Any orthogonal set of vectors is linearly independent.

○ True ○ False

2. Consider the the following system of linear equations in $\mathbb{R}^4$:

$$\begin{aligned} 2x_1 - 2x_2 + x_3 - 3x_4 &= b_1 \\ -x_1 + x_2 + 2x_4 &= b_2 \\ x_3 + x_4 &= b_3 \end{aligned}$$

(a) (7 points) Find a basis for the subspace of $\mathbb{R}^4$ consisting of the solutions to the system when $(b_1, b_2, b_3) = 0 \in \mathbb{R}^3$.

(b) (8 points) Find a basis for the subspace of $\mathbb{R}^3$ consisting of the set of vectors (b_1, b_2, b_3) such that the system is consistent.

3. Let $A = \begin{bmatrix} 0 & 1 & -1 \\ 0 & 1 & 2 \\ -1 & 0 & -1 \end{bmatrix} \in M_3(\mathbb{R})$.

   (a) (5 points) Find the characteristic polynomial of A.

(b) (5 points) Show that $A^{-1} = \frac{1}{3}(2I - A^2)$. There is an easy way to do this and a hard way to do this.

4. Let $V = P_3(\mathbb{R})$ with inner product

$$\langle p(x), q(x) \rangle = \int_0^1 p(x)q(x)dx.$$

Let $W = \mathrm{Span}(x^3 + x, x^2 - 1, x)$.

(a) (5 points) Show that $\dim W = 3$.

(b) (10 points) Find an orthogonal basis for W.

(c) (5 points) What is the dimension of $W^{\perp}$? Briefly justify your answer.

5. Let V be an inner product space.

   (a) (8 points) Let W_1 and W_2 be subspaces of V. Show that $(W_1 + W_2)^\perp = W_1^\perp \cap W_2^\perp$

(b) (7 points) Let $T\colon V \to V$ be a linear transformation such that $\langle T(u), T(v)\rangle = \langle u, v\rangle$ for all $u, v \in V$. Show that T is one-to-one.

6. If V is a vector space over a field $\mathbb{F}$, let V^* denote $\mathcal{L}(V, \mathbb{F})$, so that V^* is the space of linear transformations from V to $\mathbb{F}$. Assume that $\dim V = n$ is finite.

   (a) (4 points) Fix a vector $v \in V$. Consider the function $\phi_v \colon V^* \to \mathbb{F}$ defined by $\phi_v(f) = f(v)$. Show that ϕ_v is a linear transformation, i.e. $\phi_v \in (V^*)^* = \mathcal{L}(V^*, \mathbb{F})$.

(b) (4 points) If $\beta = \{v_1, \ldots, v_n\}$ is a basis for V, define $f_i \colon V \to \mathbb{F}$ for $1 \le i \le n$ by

$$f_i(a_1 v_1 + \cdots + a_n v_n) = a_i.$$

Show that $f_i \in V^*$.

(c) (4 points) Show that if v is not the zero vector, then there is an element $f \in V^*$ such that $\phi_v(f) \neq 0$. Part (b) may be useful!

(d) (8 points) Define $\Phi\colon V \to (V^*)^*$ by $\Phi(v) = \phi_v$. Show that Φ is a linear transformation and use (c) to conclude that Φ is an isomorphism. **Hint:** Compute $N(\Phi)$ and $\dim(V^*)^*$.

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Scratch work (second page) — DO NOT REMOVE

Scratch work (third page) — DO NOT REMOVE

Scratch work (fourth page) — DO NOT REMOVE