Math 235: Linear Algebra

Midterm Exam 1

October 21, 2014

NAME (please print legibly): Solutions

Your University ID Number: ______________________

Please circle your professor's name: **Bobkova** **Friedmann**

- **The presence of calculators, cell phones, iPods and other electronic devices at this exam is strictly forbidden.**
- **Show your work and justify your answers. You may not receive full credit for a correct answer if insufficient work is shown or insufficient justification is given.**
- **Clearly circle or label your final answers, on those questions for which it is appropriate.**

QUESTION	VALUE	SCORE
1	25	
2	10	
3	25	
4	20	
5	10	
6	10	
TOTAL	100	

1. (25 points) Let $T : M_{2\times 2}(R) \to R^2$ be a transformation defined for

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

by

$$T(A) = (a_{11} - a_{21}, -2a_{11} + 3a_{22}).$$

(a) Prove that T is linear.

Need $T(A+B) = T(A) + T(B)$ and $T(cA) = cT(A)$

Let A be as above, and $B = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}$

$$T(A+B) = T\begin{pmatrix} a_{11}+b_{11} & a_{12}+b_{12} \\ a_{21}+b_{21} & a_{22}+b_{22} \end{pmatrix} = \left((a_{11}+b_{11}) - (a_{21}+b_{21}), -2(a_{11}+b_{11}) + 3(a_{22}+b_{22}) \right)$$

$$= (a_{11} - a_{21} + b_{11} - b_{21}, -2a_{11} + 3a_{22} - 2b_{11} + 3b_{22}) =$$

$$= (a_{11} - a_{21}, -2a_{11} + 3a_{22}) + (b_{11} - b_{21}, -2b_{11} + 3b_{22}) = T(A) + T(B)$$

Let $c \in R$.

$$T(cA) = T\begin{pmatrix} ca_{11} & ca_{12} \\ ca_{21} & ca_{22} \end{pmatrix} = (ca_{11} - ca_{21}, -2ca_{11} + 3ca_{22}) =$$

$$= c(a_{11} - a_{21}, -2a_{11} + 3a_{22}) = cT(A)$$

(b) Find the kernel of T (denoted $Null(T)$ or $N(T)$ or $ker(T)$) and provide a basis for it.

$$\ker T = \{ A \in M_{2\times 2}(R) \mid T(A) = 0 \}$$

That is, $a_{11} - a_{21} = 0$ and $-2a_{11} + 3a_{22} = 0$

No conditions on a_{12}.

a_{11} is a free variable, $a_{21} = a_{11}$, $a_{22} = \frac{2}{3}a_{11}$

So $$\ker T = \left\{ \begin{pmatrix} t & s \\ t & \frac{2}{3}t \end{pmatrix} \middle| \ t, s \in R \right\} = \left\{ t\begin{pmatrix} 1 & 0 \\ 1 & 2/3 \end{pmatrix} + s\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \middle| \ t, s \in R \right\}$$

Basis: $\left\{ \begin{pmatrix} 1 & 0 \\ 1 & 2/3 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \right\}$

(c) Find the range of T (denoted $Range(T)$ or $R(T)$ or $Im(T)$) and provide a basis for it.

$$R(T) = \text{Span}\left\{T\begin{pmatrix}1&0\\0&0\end{pmatrix}, T\begin{pmatrix}0&1\\0&0\end{pmatrix}, T\begin{pmatrix}0&0\\0&1\end{pmatrix}, T\begin{pmatrix}0&0\\1&0\end{pmatrix}\right\}$$

$$= \text{Span}\{(1,-2), (0,0), (0,3), (-1,0)\} = \mathbb{R}^2$$

Basis: any pair of non-zero, non-proportional vectors.

e.g. $B = \{(1,-2), (0,3)\}$

(d) Find the matrix representation $[T]_\alpha^\beta$ where α is the standard basis of $M_{2\times 2}(R)$ and β is the standard basis of R^2.

$T: \begin{pmatrix}1&0\\0&0\end{pmatrix} \longrightarrow (1,-2)$

$\begin{pmatrix}0&1\\0&0\end{pmatrix} \longrightarrow (0,0)$

$\begin{pmatrix}0&0\\1&0\end{pmatrix} \longrightarrow (-1,0)$

$\begin{pmatrix}0&0\\0&1\end{pmatrix} \longrightarrow (0,3)$

$$[T]_\alpha^\beta = \begin{pmatrix}1&0&-1&0\\-2&0&0&3\end{pmatrix}$$

(e) Show that the rank-nullity theorem holds for T.

$\dim \ker T = 2$

$\dim R(T) = 2$

$\dim M_{2\times 2}(\mathbb{R}) = 4$

$\dim \ker T + \dim R(T) = \dim V$

$2 + 2 = 4$

2. (10 points) Mark the following statements as True or False. No explanation necessary.

(a) Let V be a vector space. If W_1 and W_2 are both subspaces of V, then $W_1 = W_2$. **True** **False**

Example: $W_1 = \text{Span}\{(1,0)\} \subseteq \mathbb{R}^2$
$W_2 = \text{Span}\{(0,1)\} \subseteq \mathbb{R}^2$

(b) The intersection of any two subspaces W_1 and W_2 of a vector space V is a subspace of V. **True** **False**

(c) If W is a subspace of V and Z is a subspace of W, then Z is a subspace of V. **True** **False**

(d) If V is a vector space having dimension n, and if S is a subset of V with n vectors, then S is linearly independent if and only if S spans V. **True** **False**

(e) Every finite dimensional vector space has a unique basis. **True** **False**

e.g. Bases for $P_1(\mathbb{R})$: $\{1, x\}$
$\{1+x, 1-x\}$

3. (25 points) Recall that if A is a 3×3 matrix, $tr(A)$ is defined as the sum of the diagonal entries of A, i.e. $tr(A) = a_{11} + a_{22} + a_{33}$.

Let $W = \{A \in M_{3\times 3}(R) \mid tr(A) = 0\}$ be a subset of the vector space of 3×3 matrices with real entries.

(Note: you may do this problem for 2×2 matrices for 60% partial credit).

(a) Prove W is a subspace of $M_{3\times 3}(R)$.

1) $\vec{0} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ $tr(\vec{0}) = 0+0+0 = 0$ so $\vec{0} \in W$

2) if $A, B \in W$ then $a_{11}+a_{22}+a_{33} = 0$ and $b_{11}+b_{22}+b_{33} = 0$

So $tr(A+B) = tr\begin{pmatrix} a_{11}+b_{11} & a_{12}+b_{12} & a_{13}+b_{13} \\ a_{21}+b_{21} & a_{22}+b_{22} & a_{23}+b_{23} \\ a_{31}+b_{31} & a_{32}+b_{32} & a_{33}+b_{33} \end{pmatrix} =$

$= a_{11}+b_{11} + a_{22}+b_{22} + a_{33}+b_{33} = (a_{11}+a_{22}+a_{33}) + (b_{11}+b_{22}+b_{33})$

$= 0+0 = 0$

So W is closed under addition

3) if $A \in W$ then $cA \in W$ for any $c \in R$ because

$tr(cA) = tr\begin{pmatrix} ca_{11} & ca_{12} & ca_{13} \\ ca_{21} & ca_{22} & ca_{23} \\ ca_{31} & ca_{32} & ca_{33} \end{pmatrix} = ca_{11} + ca_{22} + ca_{33}$

$= c(a_{11}+a_{22}+a_{33}) = c\,tr(A) = c\cdot 0 = 0$

(b) Find a basis for W.

$a_{11}+a_{22}+a_{33}=0$

All other matrix entries are unconstrained.

Let $a_{22}=t$, $a_{33}=s$ then $a_{11}=-t-s$.

Basis. $\left\{ \begin{pmatrix} 0&1&0\\0&0&0\\0&0&0 \end{pmatrix}, \begin{pmatrix} 0&0&1\\0&0&0\\0&0&0 \end{pmatrix}, \begin{pmatrix} 0&0&0\\1&0&0\\0&0&0 \end{pmatrix}, \begin{pmatrix} 0&0&0\\0&0&1\\0&0&0 \end{pmatrix}, \begin{pmatrix} 0&0&0\\0&0&0\\1&0&0 \end{pmatrix}, \begin{pmatrix} 0&0&0\\0&0&0\\0&1&0 \end{pmatrix}, \begin{pmatrix} -1&0&0\\0&1&0\\0&0&0 \end{pmatrix}, \begin{pmatrix} -1&0&0\\0&0&0\\0&0&1 \end{pmatrix} \right\}$

(c) Find the dimension of W. 8

4. (20 points) Suppose $(v_1, \ldots, v_n)$ is linearly independent set in vector space V and $w \in V$. Prove that if $(v_1 + w, \ldots v_n + w)$ is linearly dependent, then $w \in \mathrm{Span}(v_1, \ldots v_n)$.

If $(v_1+w, \ldots, v_n+w)$ is linearly dependent

then $\exists a_1, \ldots, a_n \in F$, not all a_i's zero, such that

$$\sum_{i=1}^{n} a_i(v_i+w) = 0$$

So $\sum_{i=1}^{n} a_i v_i + \sum_{i=1}^{n} a_i w = 0$

But since not all a_i's are 0 and $\{v_1, \ldots, v_n\}$ is linearly independent, we must have $\sum_{i=1}^{n} a_i v_i \neq 0$.

Hence $\sum_{i=1}^{n} a_i w \neq 0$

$\sum_{i=1}^{n} a_i w = \left(\sum_{i=1}^{n} a_i\right) w = cw$ for $c = \sum_{i=1}^{n} a_i$.

Since $cw \neq 0$, we know $c \neq 0$.

So $w = -\frac{1}{c} \sum_{i=1}^{n} a_i v_i$ and $w \in \mathrm{Span}\{v_1, \ldots, v_n\}$.

5. (10 points)

(a) Give an example of vector spaces V and W and a linear map $T : V \to W$ such that T is one-to-one but not onto.

$$T: (x, y, z) \longmapsto (x, y, z, 0)$$
$$\mathbb{R}^3 \longrightarrow \mathbb{R}^4$$

$\ker(T) = \{(0,0,0)\}$ so T is 1-1.

T is not onto because for example $(0,0,0,1) \in \mathbb{R}^4$ has no pre-image in $\mathbb{R}^3$.

(b) Give an example of vector spaces V and W and a linear map $T : V \to W$ such that T is onto but not one-to-one.

$$T: (x, y, z) \longmapsto (x, y)$$
$$\mathbb{R}^3 \longrightarrow \mathbb{R}^2$$

$\ker(T) = \{(0,0,z) \mid z \in \mathbb{R}\} \neq \{\vec{0}\}$ so T is not 1-1.

T is onto because any $w = (x,y) \in \mathbb{R}^2$ has a pre-image in $\mathbb{R}^3$, e.g. $(x, y, 0)$

6. (10 points) Let $v_1, \dots v_k, v$ be vectors in a vector space V and define $W_1 = \mathrm{Span}\{v_1, \dots v_k\}$ and $W_2 = \mathrm{Span}\{v_1, \dots, v_k, v\}$. Prove that $\dim W_1 = \dim W_2$ if and only if $v \in W_1$.

- If $v \in W_1$ then $\dim W_1 = \dim W_2$:

W_1 and W_2 are both vector spaces.

Also, W_1 is a subspace of W_2 because any element of W_1 is of the form $\sum_{i=1}^{k} a_i v_i = \left(\sum_{i=1}^{k} a_i v_i\right) + 0v$, which is an element of W_2.

Hence $\underline{\dim W_1 \leq \dim W_2}$.

Suppose $v \in W_1$. Then $v = \sum_{i=1}^{k} b_i v_i$ for some b_i's.

So any element of W_2 has the form

$$\left(\sum_{i=1}^{k} c_i v_i\right) + cv = \left(\sum_{i=1}^{k} c_i v_i\right) + c\sum_{i=1}^{k} b_i v_i$$

$$= \sum_{i=1}^{k} (c_i + cb_i)v_i \in \mathrm{Span}\{v_1, \dots, v_k\} = W_1$$

So $W_2 \subseteq W_1$ and $\underline{\dim W_2 \leq \dim W_1}$.

Therefore, $\dim W_2 = \dim W_1$.

- If $\dim W_1 = \dim W_2$ then $v \in W_1$:

We have W_1 is a subspace of W_2 as before.

By a theorem, if $\dim W_1 = \dim W_2$ then $W_1 = W_2$.

Since $v \in W_2$, we also have $v \in W_1$.