Math 235: Linear Algebra

Midterm Exam 2

November 19, 2013

NAME (please print legibly): ______________________________
Your University ID Number: ______________________________
Please circle your professor's name: Friedmann Tucker

- **The presence of calculators, cell phones, iPods and other electronic devices at this exam is strictly forbidden.**
- **Show your work and justify your answers. You may not receive full credit for a correct answer if insufficient work is shown or insufficient justification is given.**
- **Clearly circle or label your final answers, on those questions for which it is appropriate.**

QUESTION	VALUE	SCORE
1	10	
2	7	
3	18	
4	10	
5	20	
6	20	
7	15	
TOTAL	100	

1. (10 points)

Let $A = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 1 & 0 \end{pmatrix}$. Find elementary matrices $E_1, \ldots, E_n$ such that

$$E_n \ldots E_1 A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Solution. We have

$$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 1 & 0 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \longrightarrow\longrightarrow \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

The first operation is flipping the first and third rows so $E_1 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$. Next we multiply the second row by 1/2 so $E_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix}$. Finally, we add -1 time the second row to the first row, so $E_3 = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$. Then $E_3 E_2 E_1 A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$.

2. (7 points) Find the general form for the solutions to the set of linear equations:

$$x_1 + x_2 + 2x_3 = 2$$
$$x_1 + 2x_2 + x_3 = 5$$

Solution. The corresponding augmented matrix is $\begin{pmatrix} 1 & 1 & 2 & : & 2 \\ 1 & 2 & 1 & : & 5 \end{pmatrix}$. Row reducing, we get

$$\begin{pmatrix} 1 & 1 & 2 & : & 2 \\ 1 & 2 & 1 & : & 5 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 1 & 2 & : & 2 \\ 0 & 1 & -1 & : & 3 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 & 3 & : & -1 \\ 0 & 1 & -1 & : & 3 \end{pmatrix}$$

So we see that the general solution is $x_1 = -1 - 3x_3$, $x_2 = 3 + x_3$. Letting $x_3 = t$, we have the general form

$$\left\{ \begin{pmatrix} -1-3t \\ 3+t \\ t \end{pmatrix} \text{ for } t \in \mathbb{R} \right\},$$

or equivalently

$$\left\{ \begin{pmatrix} -1 \\ 3 \\ 0 \end{pmatrix} + t \begin{pmatrix} -3 \\ 1 \\ 1 \end{pmatrix} \text{ for } t \in \mathbb{R} \right\}.$$

3. (18 points) Let $A = \begin{pmatrix} 1 & 3 & 11 & 2 & 12 \\ 2 & 0 & 4 & 0 & 4 \\ 3 & 1 & 9 & 2 & 16 \\ 4 & 2 & 14 & 1 & 13 \end{pmatrix}$. The row reduced form of A is $\begin{pmatrix} 1 & 0 & 2 & 0 & 2 \\ 0 & 1 & 3 & 0 & 0 \\ 0 & 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$

(a) Find a basis for the null space of L_A. (Explain your answer.)

Solution. Looking at the row reduced form we see that the dimension of the null space will be two. We get a basis by looking at the variables that are not in columns corresponding to a leading 1 and solving for the other variables. When $x_5 = 1$ and $x_3 = 0$, we obtain $x_1 = -2$, $x_2 = 0$ and $x_4 = -5$. When $x_3 = 1$ and $x_5 = 0$, we obtain $x_1 = -2$, $x_2 = -3$, and $x_4 = 0$. Thus, we have the following basis for $N(L_A)$

$$\left\{ \begin{pmatrix} -2 \\ 0 \\ 0 \\ -5 \\ 1 \end{pmatrix}, \begin{pmatrix} -2 \\ -3 \\ 1 \\ 0 \\ 0 \end{pmatrix} \right\}$$

(b) Find a basis for the range of L_A. (Explain your answer.)

Solution. Looking a the row reduced form, we see that the first, second and fourth columns are linearly independent. Thus, a natural basis is

$$\left\{ \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix}, \begin{pmatrix} 3 \\ 0 \\ 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \\ 2 \\ 1 \end{pmatrix} \right\}$$

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(continued)

$$A = \begin{pmatrix} 1 & 3 & 11 & 2 & 12 \\ 2 & 0 & 4 & 0 & 4 \\ 3 & 1 & 9 & 2 & 16 \\ 4 & 2 & 14 & 1 & 13 \end{pmatrix} \text{ The row reduced form of } A \text{ is } \begin{pmatrix} 1 & 0 & 2 & 0 & 2 \\ 0 & 1 & 3 & 0 & 0 \\ 0 & 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

(c) Is $\left\{ \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix}, \begin{pmatrix} 3 \\ 0 \\ 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 11 \\ 4 \\ 9 \\ 14 \end{pmatrix} \right\}$ a basis $R(L_A)$? (Explain your answer.)

Solution. No, these are not linearly independent. Looking at the row reduced form we see immediately that

$$2 \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix} + 3 \begin{pmatrix} 3 \\ 0 \\ 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 11 \\ 4 \\ 9 \\ 14 \end{pmatrix}$$

(d) Is $\left\{ \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix}, \begin{pmatrix} 3 \\ 0 \\ 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 12 \\ 4 \\ 16 \\ 13 \end{pmatrix} \right\}$ a basis $R(L_A)$? (Explain your answer.)

Solution. Yes, because these are linearly independent and the size of the set if 3, which is the dimension of $R(L_A)$. To see that they are linearly independent, note that they correspond to the the first second and fifth columns in the row reduced form, and these columns are clearly independent.

4. (10 points) Suppose B is a matrix and that its row reduced form is $\begin{pmatrix} 1 & 2 & 0 & 3 & 0 \\ 0 & 0 & 1 & 2 & 0 \end{pmatrix}$.

Suppose the first column of B is $\begin{pmatrix} 3 \\ 2 \end{pmatrix}$ and the third column is $\begin{pmatrix} 1 \\ 4 \end{pmatrix}$.

(a) What is the second column of B?

Solution. The second column of B is twice the first column of B so it is $\begin{pmatrix} 6 \\ 4 \end{pmatrix}$.

(b) What is the fourth column of B?

Solution. It must be three times the first column plus two times the third so it is $\begin{pmatrix} 11 \\ 14 \end{pmatrix}$.

(c) What is the fifth column of B?

Solution. The only column that can be transformed into the zero column by row operations is the zero column itself, so the fifth column must be $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$.

5. (20 points)

(a) Let A be an $n \times n$ matrix and let S be the set of all $n \times n$ matrices C such that $CA = 0$. Prove that S is a subspace of $M_{n\times n}(\mathbb{R})$ (where $M_{n\times n}(\mathbb{R})$ is the space of all $n \times n$ matrices with coefficients in $\mathbb{R}$).

Solution. As usual, we check that S contains 0 and is closed under addition and scalar multiplication.

Clearly, if 0 is the matrix with all zero coefficients, then $0A = 0$, so $0 \in S$.

If $B, C \in S$, then $BA = 0$ and $CA = 0$, so $(B+C)A = BA+CA = 0+0$. Hence, $B+C$ is in S. This means that S is closed under addition.

If $B \in S$ and $\alpha \in F$ Is a scalar, then $BA = 0$, so $(\alpha B)A = \alpha BA = 0$. Thus, $\alpha B \in S$. This means that S is closed under scalar multiplication.

(b) Let A and B be two $n \times n$ matrices and suppose that $\mathrm{Rank}(AB) = \mathrm{Rank}(B)$. Show that $\mathrm{N}(L_A) \cap \mathrm{R}(L_B) = \{0\}$ (where L_A is the map from $\mathbb{R}^n$ to $\mathbb{R}^n$ sending v to Av and L_B is the map from $\mathbb{R}^n$ to $\mathbb{R}^n$ sending v to Bv, as usual).

Solution. By the rank-nullity theorem (what the book calls the dimension theorem), we see that if $\mathrm{Rank}(AB) = \mathrm{Rank}(B)$, then $\dim N(L_A) = \dim N(L_A L_B)$. Now, clearly, if $Bv = 0$ for $b \in \mathbb{R}^n$, then $ABv = 0$. Hence. we have $N(L_A L_B) \subseteq N(L_B)$. Since $\dim N(L_A) = \dim N(L_A L_B)$, we must then have $N(L_A) = N(L_A L_B)$. This means that $ABv = 0$ if and only if $Bv = 0$, so $Bv \in N(L_A)$ if and only if $Bv = 0$. This is equivalent to saying that $\mathrm{N}(L_A) \cap \mathrm{R}(L_B) = \{0\}$.

NOTE: Many of you said that L_A must be one-one here. That is not the case. For example, if B is zero map then we have $\mathrm{Rank}(AB) = \mathrm{Rank}(B) = 0$, regardless of what A is.

6. (20 points) In all of the questions below, $T, U : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ are linear maps. You should explain your answers carefully.

(a) True or false and explain: If UT is onto, then U is onto.

Solution. True. By a theorem we proved in class, $\operatorname{Rank}(UT) \le \operatorname{Rank} U$, but $\operatorname{Rank}(UT) = n$ because UT is onto, and $\operatorname{Rank}(U) \le n$, so $\operatorname{Rank}(U) = n$.

Alternatively: $R(UT) = U(R(T)) \subseteq U(\mathbb{R}^n) = R(U)$, but $R(UT) = \mathbb{R}^n$ so $\mathbb{R}^n \subseteq R(U)$, so $R(U) = \mathbb{R}^n$ and U is onto.

Alternatively: If UT is onto then $\forall w \in \mathbb{R}^n$, $\exists v \in \mathbb{R}^n$ such that $(UT)(v) = w$. Now let $z = T(v)$. Then $\forall w \in \mathbb{R}^n$, $\exists z \in \mathbb{R}^n$ such that $U(z) = w$, namely $z = T(v)$, so U is onto.

(b) True or false and explain: if U is onto, then UT is onto.

Solution. False. Even if U is onto, unless T is also onto, UT cannot be onto no matter what U is.

For example, let $T = 0$. Then $UT(v) = U(T(v)) = U(0) = 0$.

Another example: Let $n = 2$ and let U be the identity. Let $T(e_1) = e_1$ and $T(e_2) = e_1$. Then U is onto but $R(UT) = \operatorname{Span}\{e_1\} \neq \mathbb{R}^2$.

(c) If T is one-one, then $\operatorname{Rank}(T) = n$.

Solution. True. If T is one-to-one then $N(T) = \{0\}$. By the rank-nullity theorem, $\dim N(T) + \operatorname{Rank}(T) = n$ so $\operatorname{Rank}(T) = n - \dim N(T) = n - 0 = n$.

Alternatively: True by the theorem that says that for $T : \mathbb{R}^n \longrightarrow \mathbb{R}^n$, being one-to-one and being onto are equivalent properties (one implies the other).

(d) True or false and explain: $\text{Rank}(U+T) \leq \text{Rank}(U)$.

Solution. False. Let $U = 0$, and let T be any non-zero transformation. Then

$$\text{Rank}(U+T) = \text{Rank}(T) > 0 = \text{Rank}(U).$$

Note: some students tried to use results we proved in class about $\text{Rank}(UT)$, but this was not a question about $\text{Rank}(UT)$.

Note: the rank is NOT a linear function!! It is NOT true that $\text{Rank}(U+T) = \text{Rank}(U) + \text{Rank}(T)$.

(e) True or false and explain: if U and T are onto, then $\text{Rank}(U+T) = n$.

Solution. False. Let $U = I_n$, $T = -I_n$. Then U and T are onto. But $U + T = I_n + (-I_n) = 0$ so $\text{Rank}(U+T) = \text{Rank}(0) = 0 \neq n$.

7. (15 points) In this problem A is an $m \times n$ matrix (m rows and n columns) representing a system of m equations and n unknowns.

(a) Suppose there is a $b \in \mathbb{R}^m$ such that there is exactly one $x \in \mathbb{R}^n$ for which $Ax = b$. Show that the homogenous equation $Ax = 0$ has exactly one solution.

Solution. The solution set to $Ax = b$ is $\{s_0 + K_H\}$ where s_0 is any solution of $Ax = b$ and K_H is the solution set of $Ax = 0$. For there to be exactly one solution in the solution set of $Ax = b$, K_H can contain only one element, so $Ax = 0$ has only one solution.

Note: It does not follow that $m = n$ and that A is invertible. For example, let $m = 3$, $n = 2$, $A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}$, and $b = \begin{pmatrix} b_1 \\ b_2 \\ 0 \end{pmatrix}$. Then $Ax = b$ has a unique solution $\vec{x} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$.

Note: It does follow that for any $b \in \mathbb{R}^n$ *for which there exists a solution*, the solution is unique. But it does not follow that there exists a solution for any b; for example, with the A given in the first note, let $b = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$, then $Ax = b$ has no solutions.

Note: It does follow that $m \geq n$. If not then $m < n$ and by the rank-nullity theorem, $\dim N(L_A) = n - \text{Rank}(L_A) \geq n - m > 0$ so if $Ax = b$ has a solution, it has infinitely many solutions since $\dim K_H = \dim N(L_A) \geq 1$.

(b) Suppose now that $m > n$. Show that there is a $w \in \mathbb{R}^m$ such that there is no solution to the equation $Av = w$.

Solution. We have $L_A : \mathbb{R}^n \to \mathbb{R}^m$. Since $m > n$, L_A cannot be onto. Therefore, $\exists w \in \mathbb{R}^m$ for which there is no $v \in \mathbb{R}^n$ satisfying $L_A(v) = w$, otherwise L_A would be onto. But $L_A(v) = Av$ so if there is no v satisfying $L_A(v) = w$, there is also no v satisfying $Av = w$.

Alternatively: Rank $A \leq n$, so the row-reduced echelon form of A has at least $m - n$ rows of zeros. Suppose the row-reduced echelon form of A is $B = MA$ where M is an invertible $m \times m$ matrix. Let $z \in \mathbb{R}^m$ have all zero entries except the last one, i.e. $z = e_m$. So $Bv = z$ has no solutions. Now let $w = M^{-1}z$. Then $Av = w$ has no solutions.