Math 210, Spring 2022

Problem Set # 9

Due April 6, 2022 at 11:59pm on Gradescope

Question 1. Assume all options are European style with maturity T. A "knockout" option has payout zero if the defined event occurs.

Consider the following eight options I-VIII, where $K_1 < K_2 < K_3$.

I. K_1 call.
II. K_1 call that knocks out (i.e., has payout zero) if $S_T > K_2$.
III. K_1 call that knocks out if $S_t > K_2$ for any $0 \le t \le T$.
IV. K_1 call that knocks out if $S_T < K_1$.
V. K_1 call that knocks out if $S_t < K_1$ for any $0 \le t \le T$.
VI. (K_1, K_2) call spread (long one K_1 call, short one K_2 call).
VII. Digital call with strike K_1 and payout $K_2 - K_1$. In other words, the option whose payout at T is

$$\begin{cases} K_2 - K_1 & \text{if } S_T \ge K_1 \\ 0 & \text{if } S_T < K_1 \end{cases}$$

VIII. (K_1, K_2, K_3) call ladder (long one K_1 call, short one K_2 call, short one K_3 call)

For each of the pairs of A and B in the table below, choose the most appropriate relationship between prices at time $t \le T$ out of $=$, $\ge$, $\le$, and ?, where ? means the relationship is indeterminate.

Give justification for your answers.

Hint: Write down and compare the payouts at maturity for the options.

	A	=,≥,≤ or ?	B
(a)	I		VI
(b)	II		VI
(c)	II		III
(d)	I		IV
(e)	I		V
(f)	I		VII
(g)	VI		VII
(h)	VII		VIII
(i)	III		VIII
(j)	II		VII

Solution:

First we write out the payouts of all of the options at maturity:

i) K_1 call:

$$g(S_T) = \begin{cases} 0 & \text{if} \quad S_T \leq K_1 \\ S_T - K_1 & \text{if} \quad K_1 < S_T. \end{cases}$$

ii) K_1 call that knocks out if $S_T > K_2$:

$$g(S_T) = \begin{cases} 0 & \text{if} \quad S_T \leq K_1 \\ S_T - K_1 & \text{if} \quad K_1 < S_T < K_2 \\ 0 & \text{if} \quad K_2 \leq S_T. \end{cases}$$

iii) K_1 call that knocks out if $S_t > K_2$ for any $0 \leq t \leq T$:

$$g(S_T) = \begin{cases} 0 & \text{if} \quad S_T \leq K_1 \\ S_T - K_1 & \text{if} \quad K_1 < S_T < K_2 \quad \text{and } S_t < K_2 \\ 0 & \text{if} \quad K_2 \leq S_T \quad \text{or } S_t > K_2. \end{cases}$$

iv) K_1 call that knocks out if $S_T < K_1$:

$$g(S_T) = \begin{cases} 0 & \text{if} \quad S_T \leq K_1 \\ S_T - K_1 & \text{if} \quad K_1 < S_T. \end{cases}$$

v) K_1 call that knocks out if $S_t < K_1$ for any $0 \leq t \leq T$:

$$g(S_T) = \begin{cases} 0 & \text{if} \quad S_T \leq K_1 \quad \text{or } S_t < K_1 \\ S_T - K_1 & \text{if} \quad K_1 < S_T \quad \text{and } S_t > K_1. \end{cases}$$

vi) (K_1, K_2) call spread:

$$g(S_T) = \begin{cases} 0 & \text{if} \quad S_T \leq K_1 \\ S_T - K_1 & \text{if} \quad K_1 < S_T \leq K_2 \\ S_T - K_1 - (S_T - K_2) = K_2 - K_1 & \text{if} \quad K_2 < S_T \end{cases}$$

vii) Digital call with strike K_1 and payout $K_2 - K_1$.

$$g(S_T) = \begin{cases} 0 \quad \text{if} \quad S_T < K_1 \\ K_2 - K_1 \quad \text{if} \quad K_1 \leq S_T. \end{cases}$$

viii) (K_1, K_2, K_3) call ladder.

$$g(S_T) = \begin{cases} 0 & \text{if} \quad S_T \le K_1 \\ S_T - K_1 & \text{if} \quad K_1 < S_T \le K_2 \\ S_T - K_1 - (S_T - K_2) = K_2 - K_1 & \text{if} \quad K_2 < S_T \le K_3 \\ K_3 + K_2 - K_1 - S_T & \text{if} \quad K_3 < S_T. \end{cases}$$

a) I and VI: If $S_T \le K_2$ then we have I=VI. If $S_T > K_2$ then $K_2 - K_1 < S_T - K_1$ and so we have I $\ge$ VI.

b) II and VI: If $S_T < K_2$ then we have II=VI. If $S_T > K_2$ then $0 \le K_2 - K_1$ and so we have II $\le$ VI.

c) II and III: In all cases II $\ge$ III.

d) I and IV: In this case I = IV.

e) I and V: In this case I $\ge$ V.

f) I and VII: If $S_T < K_1$ then I=VII. However if $K_1 \le S_T$ then they cannot be compared.

g) VI and VII: If $S_T \le K_1$ then VI=VII. If $K_1 < S_T \le K_2$ then $S_T - K_1 \le K_2 - K_1$ and if $K_2 \le S_T$ then VI=VII. Thus we have VI $\le$ VII.

h) VII and VIII: If $S_T \le K_3$ then VII $\ge$ VIII. If $K_3 \le S_T$ then VII $\ge$ VIII.

i) III and VIII: If $S_t > K_2$ but $K_3 < S_T$ then VIII $< 0 <$ III. However if $K_1 < S_T \le K_2$ but $S_t > K_2$ then VIII $>$ III. Thus they cannot be compared.

j) II and VII: In all cases II $\le$ III.

	A	=,≥,≤ or ?	B
(a)	I	$\geq$	VI
(b)	II	$\leq$	VI
(c)	II	$\geq$	III
(d)	I	$=$	IV
(e)	I	$\geq$	V
(f)	I	?	VII
(g)	VI	$\leq$	VII
(h)	VII	$\geq$	VIII
(i)	III	?	VIII
(j)	II	$\leq$	VII

Theorem 0.1 (Convexity of Call Options)**.** Let $K_1 < K_2$, and let $\lambda \in (0,1)$ be a constant. Define

$$K^* := \lambda K_1 + (1-\lambda)K_2.$$

(Think of this as a weighted average of K_1 and K_2; if $\lambda = 1/2$ it's the usual average.) Then

$$C_{K^*}(t,T) \leq \lambda C_{K_1}(t,T) + (1-\lambda)C_{K_2}(t,T). \tag{1}$$

In words, this says that $C_K(t,T)$ is a concave up function of the strike price K.

Question 2. This question will have you prove the above theorem two different ways.

a) First, the calculus proof: Show that $\dfrac{\partial^2}{\partial K^2}C_K(t,T)$ is positive. Hint: you already did the hard part of this in question 2(b) on homework 8. Now just interpret that result. Is the price of a digital call an increasing or decreasing function of the strike price?

b) Second, prove that Equation (1) holds directly by considering the payout of the following call butterfly:

$$\begin{cases} \text{Long } \lambda \ K_1\text{-calls} \\ \text{Long } (1-\lambda) \ K_2\text{-calls} \\ \text{Short one } K^*\text{-call} \end{cases}$$

Solution:

(a) From homework 8, we have

$$-\frac{\partial}{\partial K}C_K(t,T) = V(t,T) \Leftrightarrow -\frac{\partial^2}{\partial^2 K}C_K(t,T) = \frac{\partial}{\partial K}V(t,T),$$

where $V(t,T)$ is the value of a digital call with maturity T and strike price K. Since the value of a digital call *decreases* as K increases, we know its partial derivative with respect to K is negative. Thus, the second derivative of $C_K(t,T)$ with respect to K is positive, i.e. $C_K(t,T)$ is convex as a function of K.

(b) The payout of this call butterfly is as follows:

$$\begin{cases} 0, \text{ if } S_T \leq K_1 \\ \lambda(S_T - K_1), \text{ if } K_1 \leq S_T < K^* \\ (1+\lambda)(K_2 - S_T), \text{ if } K^* \leq S_T < K_2 \\ 0, \text{ if } K_2 \leq S_T \end{cases}$$

Note that in all cases, the payout is ≥ 0. Since the payout of this portfolio is simply the sum of the values of each element of the butterfly evaluated at time T, we have

$$\lambda C_{K_1}(T,T) - C_{K^*}(T,T) + (1-\lambda)C_{K_2}(T,T) \geq 0$$

thus, by Monotonicity, we have

$$\lambda C_{K_1}(t,T) - C_{K^*}(t,T) + (1-\lambda)C_{K_2}(t,T) \geq 0,$$

$$\lambda C_{K^*}(t,T) \leq C_{K_1}(t,T) + (1-\lambda)C_{K_2}(t,T).$$

Question 3. Consider a stock paying no income which has current price $S_0 = 100$, and each year, the new stock price is either 30% above, or 15% below the previous price. Assume a fixed 5% annually compounded zero rate.

Define a function

$$f(K) = C_K(0,2),$$

i.e. the price of a European K-call on this stock with two year maturity.

a) Express $f(K)$ as a piecewise linear function. Simplify as much as you can.

b) Plot the function f on the domain $K \in [50, 200]$.

Solution: a) We have $u = .3$ and $d = -.15$. The risk neutral probability is

$$p^* = \frac{.05 + .15}{.3 + .15} = 4/9.$$

At year two, the possible stock prices are $S_2 = 100(1.3)^2 = 169$ with probability $(4/9)^2$, $S_2 = 100(1.3 \cdot .85) = 110.50$ with probability $2(4/9)(5/9)$, and $S_2 = 100(.85)^2 = 72.25$ with probability $(5/9)^2$.

If $K \leq 72.25$, the call will always be exercised, and pays out $S_2 - K$ in all three outcomes. Then

$$f(K) = Z(0,2)\left((169-K)\cdot(4/9)^2 + (110.5-K)\cdot 2(4/9)(5/9) + (72.25-K)\cdot(5/9)^2\right) = 100 - .907K.$$

If $72.25 \leq K \leq 110.5$, the call will not be exercised if $S_2 = 72.25$. Then

$$f(K) = Z(0,2)\left((169-K)\cdot(4/9)^2 + (110.5-K)\cdot 2(4/9)(5/9) + (0)\cdot(5/9)^2\right) = 79.7738 - .6271K$$

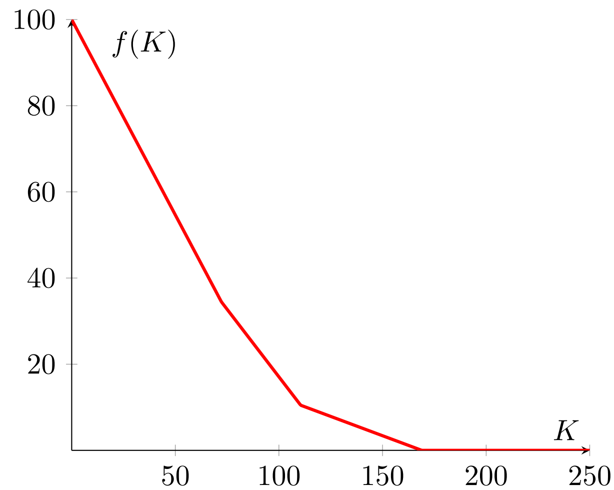
If $110.5 \le K \le 169$, the call will only be exercised if $S_2 = 169$. Then

$$f(K) = Z(0,2)\left((169-K)\cdot(4/9)^2 + (0)\cdot 2(4/9)(5/9) + (0)\cdot(5/9)^2\right) = 30.2791 - .1792K$$

Finally, if $K > 169$, the call will not be exercised, and thus has value zero. In total,

$$f(K) = \begin{cases} 100 - .907K & K \le 72.25 \\ 79.7738 - .6271K & 72.25 \le K \le 110.50 \\ 30.2791 - .1792K & 110.50 \le K \le 169 \\ 0 & 169 \le K \end{cases}$$

b) The plot of the above function is



Note that even thought this is a discrete model, we can see the convexity of $f(K) = C_K(0,2)$.