Math 210, Fall 2021

Problem Set # 8

Due March 30, 2022 at 11:59pm on Gradescope

Question 1. Tesla stock (TSLA) is currently trading at \$1200. You think it is overvalued and decide to short the stock. You're worried, however, that if the stock instead keeps going up, you could owe an arbitrarily large amount of money on this short position.

(a) If you want to keep the short position but cap your possible losses at at most \$300 a year from now, what option(s) should you add to your portfolio?

(b) Suppose instead that your broker will not sell you a short position in the stock, and is only offering (long or short) call options on the stock. Construct a portfolio whose payout is positive if and only if one year from now the stock price has dropped to between \$600 and \$800.

(c) Now suppose your broker is only offering put options. Construct a portfolio with the same payout as that in (b) using only puts.

Solution:

(a) Go long a \$1500-call on TSLA. If the stock stays below \$1500, you don't exercise the call and the payout is the same as just the short. If it goes above \$1500, you exercise the call, and then pay \$1500 to buy the stock and pay off the short position.

(b) A symmetric call butterfly will accomplish this:

- Long one \$600-call
- Long one \$800-call
- Short two \$700-calls

(c) We can construct a butterfly out of put options as well:

- Long one \$600-call
- Long one \$800-call
- Short two \$700-calls

Question 2. a) Draw a payout graph for the following two call spread portfolios:

(i) long one K-call and short one $(K+1)$-call.

(ii) long two K-calls and short two $(K + .5)$-calls.

b) A digital call with strike price K and maturity T pays out 1 if $S_T \geq K$ and 0 if $S_T < K$. By constructing a series of portfolios of call spreads and taking limits, prove that the price at time t of a digital all, with strike K^* and payout 1, is given by

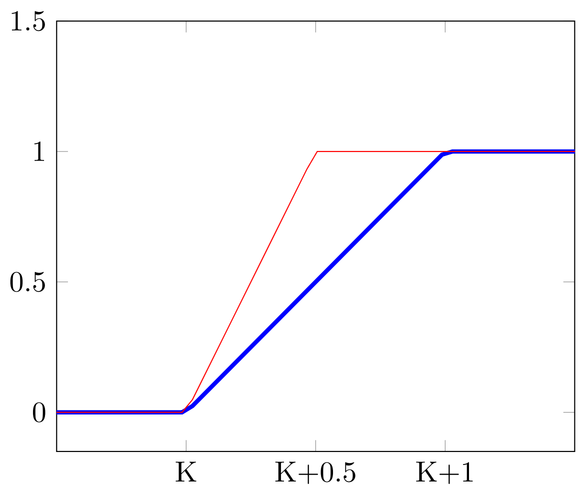
$$-\frac{\partial}{\partial K} C_K(t,T)\big|_{K^*},$$

where $\big|_{K^*}$ means the function is evaluated at $K = K^*$. *Hint: recall the limit definition of the derivative from Calc I.*

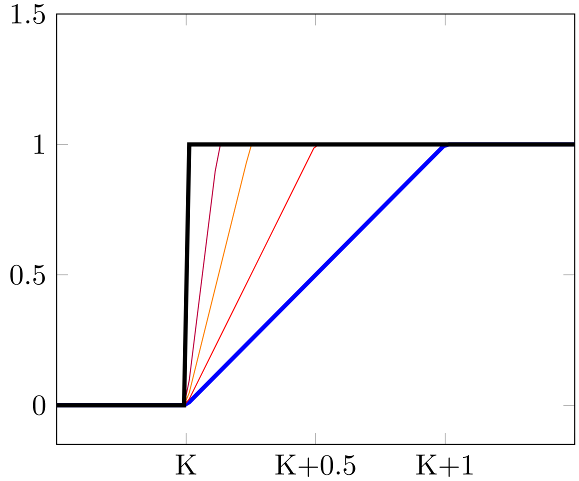
c) A digital put with strike price K and maturity T pays out 1 if $S_T \leq K$ and 0 if $S_T > K$. Write down the equivalent formula for a digital put option in terms of put prices.

d) By examining the payout profile, derive a put-call parity relationship for the digital call and digital put.

Solution:

(a)


(b) More generally, we can consider the sequence of call spread portfolios where portfolio n consists of long n K calls and short n $(K + \frac{1}{n})$-calls. As $n \to \infty$, the payout of these portfolios approaches that of a digital call:



In math terms, taking the limit of the payout of these portfolios,

$$\lim_{n\to\infty} n\left(C_K(t,T) - C_{K+\frac{1}{n}}(t,T)\right) = \text{ the price of a digital call.}$$

Letting $h = \frac{1}{n}$,

$$\lim_{n\to\infty} n\left(C_K(t,T) - C_{K+\frac{1}{n}}(t,T)\right) = \lim_{h\to 0} \frac{C_K(t,T) - C_{K+h}(t,T)}{h}$$

$$= -\lim_{h\to 0} \frac{C_{K+h}(t,T) - C_K(t,T)}{h} = -\frac{\partial}{\partial K} C_K(t,T)\big|_{K^*}.$$

(c) For puts, we can approximate a digital put using portfolios of the form short n K puts, and long n $(K + \frac{1}{n})$. Note that the long and short strike price are switched from above. This switch means that we can derive the same relationship like in (b), except without the negative sign. The price of a digital put is then

$$\frac{\partial}{\partial K} P_K(t,T)\big|_{K^*}$$

(d) Put-Call parity says $C_K(t,T) - P_K(t,T) = V_K(t,T)$. Change the sign of both sides to get $-C_K(t,T) + P_K(t,T) = -V_K(t,T)$. Take the K derivative of both sides, and use (b) and (c) above:

$$(\text{digital } K\text{-call price}) + (\text{digital } K\text{-put price}) = -\frac{\partial}{\partial K} V_K(t,T)|_K$$

Recall that $V_K(t,T) = (F(t,T) - K)Z(t,T)$. The K derivative is $-Z(t,T)$, since the forward price is a constant independent of K. The relation we get is

$$(\text{digital } K\text{-call price}) + (\text{digital } K\text{-put price}) = Z(t,T).$$

There is alternatively a much easier way to see this: If you have a portfolio long one digital K-call and long one digital K-put, it pays out 1 if S_T is below K, and 1 if S_T is above K. (We can ignore what happens if $S_T = K$, since the odds of this are very small.) A guaranteed payout of 1 at time T is the same as a ZCB with maturity T.

Remark 1. Like for regular calls, the price of a digital call decreases as the strike price increases. Using (b), this means that the second K-derivative of $C_K(t,T)$ is positive, that is, $C_K(t,T)$ is a concave up function of K. We proved this a different way in class using a call butterfly; here's a calculus proof that says the same thing.

Similarly, since a digital put price increases as the strike price increases, $P_K(t,T)$ is also concave up.

Question 3. We showed in class that the price at time $t \le T$ of a K call on a stock paying no dividends satisfies

$$C_K(t,T) \ge \max(S_t - KZ(t,T), 0).$$

a) Use the above bound to prove that if $t \le T_1 \le T_2$,

$$C_K(t,T_2) \ge C_K(t,T_1).$$

b) Does the same hold for puts? That is, prove or find a counterexample to the statement

$$P_K(t,T_2) \ge P_K(t,T_1) \text{ for } t \le T_1 \le T_2.$$

Solution:

(a) At time T_1, we know that $C_K(T_1,T_1) = (S_{T_1} - K)^+$. From part (a),

$$C_K(T_1,T_2) \ge \max(S_{T_1} - KZ(T_1,T_2), 0) \ge \max(S_{T_1} - K, 0) = C_K(T_1,T_1),$$

Since $Z(T_1,T_2) \le 1$. Then by monotonicity, $C_K(t,T_2) \ge C_K(t,T_1)$ for any $t \le T_1$.

(b) No. Suppose we're at time $t = T_1$, and the stock is currently worth $S_{T_1} = 0$. Then $P_K(T_1, T_1) = K$. The payout of the maturity T_2 put is at most K, since the stock price can't drop below zero. Discounted to present (T_1) value, the payout has value at most $KZ(T_1, T_2) < K$, and so $P_K(T_1, T_2)$ must have a lower value than $K = P_K(T_1, T_1)$ in this example.

Question 4. An asymmetric call butterfly with strikes (K_1, K_2, K_3) is a portfolio consisting of long 1 K_1 call, short 2 K_2 calls, and long 1 K_3 call, all with maturity T. Find the payout function at maturity T for a call butterfly with strikes $(70, 105, 110)$. Graph the payout at maturity as a function of the stock price.

Solution: The payout function at maturity is

$$g(S_T) = \begin{cases} 0 & \text{if} \quad S_T \leq 70 \\ S_T - 70 & \text{if} \quad 70 < S_T \leq 105 \\ S_T - 70 - 2(S_t - 105) = -S_T + 140 & \text{if} \quad 105 < S_T \leq 110 \\ -S_T + 140 + S_T - 110 = 30 & \text{if} \quad 110 < S_T. \end{cases}$$

Plot: