Math 210, Spring 2022

Problem Set # 7

Due March 23, 2021 at 11:59pm on Gradescope

Question 1. The futures price on gold takes the following values over the next five days:

Day	$\Phi(t,T)$	MTM	Margin@open	payment	Margin@close
0	2100				
1	2000				
2	2050				
3	1900				
4	2000				
5	2150				

Suppose the initial margin requirement is 10%, with maintenance margin requirement 5%. Ignoring interest, how much money do you make or lose if you enter one futures contract, then:

a) Exit the futures contract the first time you get margin called.

b) Post additional margin when required and exit the futures contract on day 5.

Solution: See table below, corresponding to (b). In (a), on Day 3, when you are margin called, you do not deposit the additional 180 (in order to get your margin up to 10% of 1900), instead you exit the contract, making $-210 + 10 = -200$, i.e. losing 200. In (b), you do not exit the contract on day 3, thus continue payments through day 5. The sum of your payments is 50. Note that remaining in the futures contract paid off despite the initial negative payment of 180.

Day	$\Phi(t,T)$	MTM	Margin@open	payment	Margin@close
0	2100	$\sim$	0	-210	210
1	2000	-100	110	0	110
2	2050	$+50$	160	0	160
3	1900	-150	10	-180	190
4	2000	$+100$	290	$+90$	200
5	2150	$+150$	350	$+350$	0

Question 2. Assume the continuously compounded interest rate has constant value 12%. The table below is for a futures contract maturing on day 6 with delivery price equal to the futures price. The underlying asset is a stock paying no income. The S_t column gives the stock price on each day. The $\Phi(t,T)$ column gives the futures price on each day. Assume $T =$ day 6. The MTM column lists the mark-to-market payments. The interest column lists the interest that will be accrued on the mark-to-market payment by the maturity date.

Fill in the table. Give at least four decimal places.

day	S_t	$\Phi(t,T)$	MTM	interest
0	1900			
1	2000			
2	2100			
3	2200			
4	2000			
5	2100			
6	2050			
		sum:		

Hint: Use Mathematica or a spreadsheet (i.e. Excel) for the calculations.

Solution: Assuming the interest rates are constant we have $\Phi(t,T) = F(t,T) = S_t e^{r(T-t)}$ where $r = 12\%$. The MTM payments are $\Phi(t+i\Delta,T) - \Phi(t,T)$ where $\Delta = 1/365$. Each MTM payment accrues interest with rate $r = 12\%$. Note that the interest is expressed per annum. As an example we compute the payments on day 1:

$$\Phi(1/365, 5/365) = S_{1/365} e^{.12(5-1)/365} = 1903.1259$$

The MTM payment on day 1 is

$$\Phi(1/365, 6/365) - \Phi(0/365, 6/365) = 2003.2904 - 1903.7516 = 99.5387.$$

The interest accrued from the MTM payment by the maturity date is

$$99.5387(e^{0.12(6-1)/365} - 1) = 0.1638.$$

The remaining values are given in the table below.

day	S_t	$\Phi(t,T)$	MTM	interest
0	1900	1903.7516	0	
1	2000	2003.2904	99.5387	0.1638
2	2100	2102.7635	99.4731	0.1309
3	2200	2202.1709	99.4075	0.0981
4	2000	2001.3155	-200.8554	-0.1321
5	2100	2100.6905	99.3750	0.0327
6	2050	2050.00	-50.6905	0
		sum:	146.2484	0.2933

Question 3. (10pts Bonus, not required) As we defined in class, a FRA with maturity T, term length α and fixed rate K exchanges payments of $\alpha L_T[T, T+\alpha]$ and αK at time $T+\alpha$. The *forward libor rate* $L_t[T, T+\alpha]$ is the fixed rate K which makes this contract have value zero at time t.

Now consider a FRA which exchanges payments of $\alpha L_T[T, T+\alpha]$ and αK just like above, but does so at time T instead of time $T+\alpha$. Is the forward libor rate for the FRA which pays at time T higher or lower than the forward libor rate for the FRA which pays at time $T+\alpha$?

Solution:

The first FRA has payout $\alpha(L_T[T, T+\alpha] - K)$ at time $T+\alpha$. The second pays out $\alpha(L_T[T, T+\alpha] - K)$ at time T. Whatever this value is, invest it at the Libor rate from time T to $T+\alpha$ so that we can compare the value of these two FRAs at the same time $(T+\alpha)$. The difference in payouts is:

$$
\begin{aligned}
(1+\alpha L_T[T, T+\alpha])\alpha(L_T[T, T+\alpha] - K) - \alpha(L_T[T, T+\alpha])) \\
= \alpha^2 L_T[T, T+\alpha]\,(L_T[T, T+\alpha] - K) \quad (1)
\end{aligned}
$$

If a derivative with this payout has positive present value, the first is worth more. That's what we'll show.

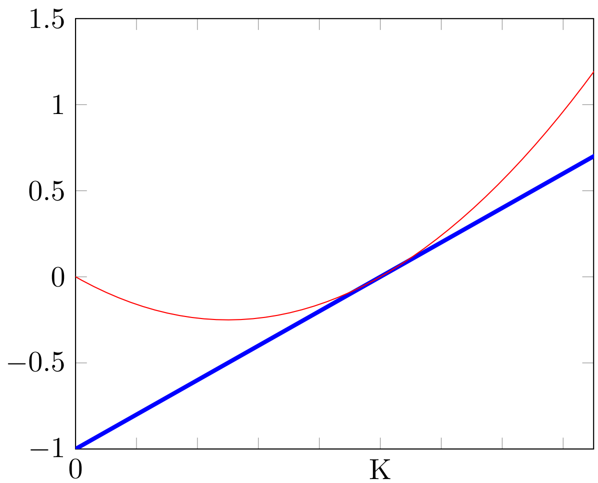
Suppose K is equal to the forward libor rate for the normal FRA. That means, by definition, that the FRA with payout $\alpha(L_T[T, T+\alpha] - K)$ at time $T+\alpha$ has present value equal to zero. Let portfolio A consist of this single, normal FRA. Compare this to Portfolio B, which consists of long (K/α) FRAs which pay out at T and short (K/α) FRAs which pay out at $T+\alpha$. The value of Portfolio B at time $T+\alpha$ is just (K/α^2) times Equation (1), i.e.

$$K\alpha L_T[T,T+\alpha]\left(L_T[T,T+\alpha]-K\right).$$

To see more clearly what's happening, note that α and K are fixed, and $L_T[T,T+\alpha]$ is a random variable. We'll write X for this random variable to make the equation look cleaner. Then Portfolio A is worth $\alpha(X-K)$ at time $T+\alpha$, and Portfolio B is worth $\alpha KX(X-K)$. And, no matter what X is,

$$\alpha KX(X-K)\geq\alpha(X-K),$$

with equality if and only if $X=K$. The easiest way to see that this inequality holds is to note that the left side is a parabola, and the right side is its tangent line. Here's the graph:



Since, by definition, Portfolio A has present value zero, Portfolio B must have positive present value. Thus with the same fixed rate K, the FRA which pays out at T is worth more. To make it have zero present value, the forward libor rate will have to be higher than the normal forward libor rate.

You can also see this intuitively from equation (1). The payout can be positive or negative, depending on whether the libor rate ends up above or below the fixed rate, just like for a normal FRA. But with the extra libor rate term in front, positive payouts get multiplied by larger libor rates, and negative payouts get multiplied by smaller libor rates. Since you gains are scaled up while your losses are scaled down (in a relative sense) this new derivative must be worth comparatively more!