Math 210, Spring 2022

Problem Set # 6

Due **Friday**, March 4, 2022 at 11:59pm on Gradescope

Question 1. Suppose the current one-year euro swap rate $y_0[0,1]$ is 1.74%, and the two-year and three-year swap rates are 2.24% and 2.55% respectively. Euro swap rates are quoted with annual payments and 30/360 daycount (thus $\alpha = 1$).

a) Use bootstrapping to calculate $Z(0,1), Z(0,2)$ and $Z(0,3)$ and obtain $P_0[0,3]$ the present value of a three-year annuity paying €1 per year.

b) Recall that the annually compounded zero rate for maturity T is the rate r such that $Z(0,T) = (1+r)^{-T}$. Calculate the one-year, two-year and three-year zero rates and compare them to the swap rates. For upward sloping yield curves, that is, when $y_0[0,T_2] > y_0[0,T_1]$ for $T_2 > T_1$, will zero rates be higher or lower than swap rates?

c) An approximate short-cut sometimes used on trading desks to calculate the present value of, say, a three-year annuity is to discount each payment at the three-year swap rate. In other words, it is assumed that $Z(0,n) = (1 + y_0[0,3])^{-n}$ for $n = 1, 2, 3$. this is often called IRR (internal rate of return) discounting. Calculate the error in valuing the annuity in (a) this way.

d) Calculate the price of a three-year fixed rate bond of notional €1 and annual coupons of 2.55% using the ZCB prices calculated in (a), and verify this equals the price obtained via IRR discounting at a rate of 2.55%

Solution:

a) We use that the forward swap rate at $t \le T_0$ for a swap from T_0 to T_n is

$$y_t[T_0, T_n] = \frac{Z(t,T_0) - Z(t,T_n)}{P_t[T_0,T_n]} = \frac{Z(t,T_0) - Z(t,T_n)}{\sum_{i=1}^{n} \alpha Z(t,T_i)}$$

For positive integer k and $\alpha = 1$, we have

$$y_0[0,k] = \frac{Z(0,0) - Z(0,k)}{\sum_{i=1}^{k} Z(0,i)} = \frac{1 - Z(0,k)}{\sum_{i=1}^{k} Z(0,i)}.$$

Solving for $Z(0,k)$ gives

$$Z(0,k) = \frac{1 - y_0[0,k] \sum_{i=1}^{k-1} Z(0,i)}{1 + y_0[0,k]}.$$

Therefore

$$Z(0,1) = \frac{1}{1 + y_0[0,1]} = \frac{1}{1 + 0.0174} = 0.982898\ldots.$$

$$Z(0,2) = \frac{1 - y_0[0,2]Z(0,1)}{1 + y_0[0,2]} = \frac{1 - (0.0224)Z(0,1)}{1 + 0.0224} = 0.956556\ldots.$$

$$Z(0,3) = \frac{1 - y_0[0,3](Z(0,1) + Z(0,2))}{1 + y_0[0,3]} = \frac{1 - (0.0255)(Z(0,1) + Z(0,2))}{1 + 0.0255} = 0.926908\ldots.$$

$$P_0[0,3] = Z(0,1) + Z(0,2) + Z(0,3) = 2.866362\ldots$$

b) The T-year annual zero rate $r = r[0,T]$ satisfies

$$Z(0,T) = (1+r)^{-T}$$

and so

$$r = r[0,T] = Z(0,T)^{-1/T} - 1.$$

The one-year annual zero rate is

$$r[0,1] = Z(0,1)^{-1} - 1 = 0.0147$$

The two-year annual zero rate is

$$r[0,2] = Z(0,2)^{-1/2} - 1 = 0.022456\ldots.$$

The three-year annual zero rate is

$$r[0,3] = Z(0,3)^{-1/3} - 1 = 0.025623\ldots.$$

We are given that:

The one-year swap rate is

$$y_0[0,1] = 0.0147$$

The two-year swap rate is

$$y_0[0,2] = 0.0224$$

The three-year swap rate is

$$y_0[0,3] = 0.0255$$

Thus the zero rates are higher than the swap rates.

Remark about (b): The graph of $y_0[0,T]$ versus T is called a yield curve. The yield curve is upward sloping if $y_0[0,T_1] > y_0[0,T_2]$ whenever $T_1 > T_2$. In our particular case, we had $y_0[0,1] > y_0[0,2] > y_0[0,3]$. This does not mean the yield curve is upward sloping; but it is consistent with upward sloping. And we saw that, in our particular case, the (annually compounded) zero rates are higher than the swap rates. In general, if the yield curve is upward sloping, the zero rates will be higher than swap rates.

c) With IRR discounting, the annuity from (a) (which pays 1 each at time 1, 2, and 3) is

$$\begin{aligned} P_0^{\mathrm{IRR}}[0,3] &= Z^{\mathrm{IRR}}(0,1) + Z^{\mathrm{IRR}}(0,2) + Z^{\mathrm{IRR}}(0,3) \\ &= (1+y_0[0,3])^{-1} + (1+y_0[0,3])^{-2} + (1+y_0[0,3])^{-3} \\ &= (1+0.0255)^{-1} + (1+0.0255)^{-2} + (1+0.0255)^{-3} \\ &= 2.853262\ldots \end{aligned}$$

The error in using IRR discounting for this valuation is

$$\text{error} = |P_0[0,3] - P_0^{\mathrm{IRR}}[0,3]| = |(2.866362\ldots) - (2.853262\ldots)| = 0.0131\ldots.$$

d) Using the results of (a), the current price of a three-year fixed rate bond of notional 1 and annual coupons $c = y_0[0,3] = 0.0255$ is

$$cZ(0,1) + cZ(0,2) + cZ(0,3) + Z(0,3) = 1 \quad \text{exactly.}$$

Question 2. We have assumed that the payment dates for the fixed and floating legs of a swap are the same. However, in practice, the payment frequencies may differ. For example, in the US swap market, the fixed leg usually has semi-annual payments ($\alpha = 0.5$) and the floating leg has quarterly payments ($\alpha = 0.25$).

a) Draw a diagram similar to Figure 4.1 on page 34 of Blyth (also in the 10/04 lecture notes) for a swap where the fixed payments are semi-annual and floating payments are quarterly.

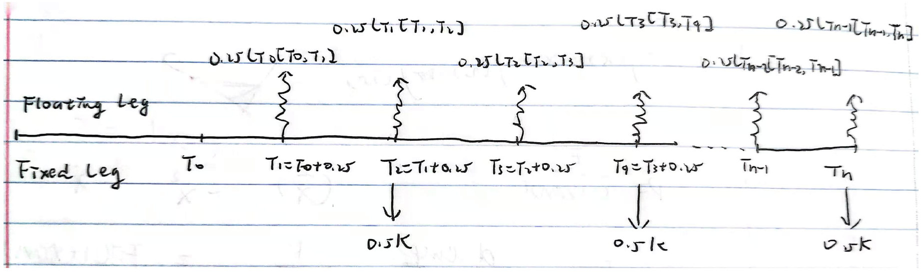
b) Consider a swap from T_0 to T_n with fixed rate K. Suppose the term length for the floating leg is $\alpha(\mathrm{FL})$ and the term length for the fixed leg is $\alpha(\mathrm{FXD})$. Write down a formula for the value of the swap $V_K^{SW}(t)$ (where $t \le T_0$) in terms of ZCB prices.

c) This continues (b). Does the value of the swap depend on $\alpha(\mathrm{FL})$? Does it depend on $\alpha(\mathrm{FXD})$?

d) This continues (b). As usual, the swap rate $y_t[T_0, T_n]$ is the special fixed rate such that $V^{SW}_{y_t[T_0,T_n]}(t) = 0$. Is the swap rate larger if the fixed leg payments are quarterly or if they are annually?

Solution:

a) Graph:



b) The value of the floating leg is $Z(t, T_0) - Z(t, T_n)$, and the value of the fixed leg is $\alpha(\text{FXD})K \sum_{i=1}^{n} Z(t, T_i)$. Thus the total value is

$$VTSW_K(t) = Z(t, T_0) - Z(t, T_n) - \alpha(\text{FXD})K \sum_{i=1}^{n} Z(t, T_i).$$

c) We can see from the answer to (b) that it depends only on $\alpha(\text{FXD})$, and not on $\alpha(\text{FL})$.

d) The forward swap rate is larger when fixed leg payments are annual as opposed to quarterly. Consider the equation for the forward swap rate:

$$y_t[T_0, T_n] = \frac{Z(t, T_0) - Z(t, T_n)}{\alpha \sum_{i=1}^{n} Z(t, T_i)}$$

The numerator stays the same no matter the payment frequency. If the payments are quarterly they will start coming earlier than if they are annually, which means that the same total payments each year will be discounted less. This makes the denominator bigger when payments are quarterly.