Math 210, Spring 2022

Problem Set # 5

Due February 23 at 11:59pm on Gradescope

Question 1. A bank has borrowing needs at time $T > 0$. Show that by combining an FRA trade today with a libor loan at time T, the bank can today lock in its interest cost for the period T to $T + \alpha$. Does the borrowing bank need to buy or sell the FRA to do this? What is the fixed rate that the bank locks in?

Solution:

The bank should buy 1 FRA at the forward rate and receive a libor loan of 1 at time T. The payout of the FRA at time $T + \alpha$ is $\alpha(L_T[T, T + \alpha] - L_t[T, T + \alpha])$. The payout of the loan at time $T + \alpha$ is $-(1 + \alpha L_T[T, T + \alpha])$. Combining these 2 transactions the payout at time $T + \alpha$ will be $-(1 + \alpha L_t[T, T + \alpha])$. Thus if the bank buys 1 FRA at the forward libor rate and receives a libor loan of 1 at time T then it can lock in its interest cost for the period T to $T + \alpha$. The bank receives the fixed rate $L_t[T, T + \alpha] = \frac{Z(t,T) - Z(t,T+\alpha)}{\alpha Z(t,T+\alpha)}$.

Question 2. Suppose $t \leq T_1 \leq T_2 \leq T_3$, where t is the current time, and $\Delta > 0$. Recall that $Z(T_1, T_2)$ is the price at time T_1 of a ZCB with maturity T_2 and $F(T_1, T_2, T_3)$ is the forward price at time T_1 for a forward contract with maturity T_2 on a ZCB with maturity T_3. Assume all interest rates are non-negative.

a) For each of the pairs of A and B in the table, choose the most appropriate relationship out of $\geq, \leq, =$, ?, where ? means the relationship is indeterminate. Give brief reasoning.

	A	$\geq, \leq, =$, ?	B
(i)	$Z(t, T_1)$		1
(ii)	$Z(T_1, T_1)$		1
(iii)	$Z(t, T_2)$		$Z(t, T_3)$
(iv)	$Z(T_1, T_2)$		$Z(T_1, T_3)$
(v)	$Z(T_1, T_3)$		$Z(T_2, T_3)$
(vi)	$Z(T_1, T_1 + \Delta)$		$Z(T_2, T_2 + \Delta)$
(vii)	$F(t, T_1, T_2)$		$F(t, T_1, T_3)$
(viii)	$F(t, T_1, T_3)$		$F(t, T_2, T_3)$
(ix)	$\lim_{T\to\infty} Z(t, T)$		0

Hint: Remember that at current time t, $F(t, \cdot, \cdot)$ is known but $Z(T, \cdot)$ is a random variable.

Solution:

	A	$\geq, \leq, =, ?$	B
(i)	$Z(t,T_1)$	$\leq$	1
(ii)	$Z(T_1,T_1)$	$=$	1
(iii)	$Z(t,T_2)$	$\geq$	$Z(t,T_3)$
(iv)	$Z(T_1,T_2)$	$\geq$	$Z(T_1,T_3)$
(v)	$Z(T_1,T_3)$	?	$Z(T_2,T_3)$
(vi)	$Z(T_1,T_1+\Delta)$	?	$Z(T_2,T_2+\Delta)$
(vii)	$F(t,T_1,T_2)$	$\geq$	$F(t,T_1,T_3)$
(viii)	$F(t,T_1,T_3)$	$\leq$	$F(t,T_2,T_3)$
(ix)	$\lim_{T\to\infty} Z(t,T)$	$=$	0

i) Receiving 1 in the future is worth less than receiving 1 now.

ii) At future time T_1 the value of receiving 1 is 1.

iii) Receiving 1 at the later date T_3 is worth less than receiving 1 at the earlier date T_2.

iv) Since $T_2 \leq T_3$ the promise of receiving 1 at the earlier time T_2 is worth more than the promise of receiving 1 at the later time T_3.

v) We cannot compare the forward rate between T_1 and T_3 to the forward rate between T_2 and T_3 since interest rates can fluctuate over time.

vi)

$$F(t,T_1,T_2) = \frac{Z(t,T_2)}{Z(t,T_1)}$$
$$F(t,T_1,T_3) = \frac{Z(t,T_3)}{Z(t,T_1)}$$

Since $Z(t,T_2) \geq Z(t,T_3)$ the result follows.

vii)

$$F(t,T_2,T_3) = \frac{Z(t,T_3)}{Z(t,T_2)} \geq \frac{Z(t,T_3)}{Z(t,T_1)} = F(t,T_1,T_3)$$

where we used the fact $Z(t,T_1) \geq Z(t,T_2)$.

viii) Never receiving one dollar has zero value.

b) What can you say about interest rates between T_1 and T_2 if

i) $Z(t,T_1) = Z(t,T_2)$?

ii) $Z(t, T_1) > 0$ and $Z(t, T_2) = 0$?

Solution:

i) Since $F(t, T_1, T_2) = \frac{Z(t,T_2)}{Z(t,T_1)}$ we get $F(t, T_1, T_2) = 1$. The forward rate f_{12} between T_1 and T_2 it is related to the ZCB price by

$$F(t, T_1, T_2) = \frac{1}{(1 + f_{12})^{T_2 - T_1}},$$

thus $f_{12} = 0$.

ii) In this case $F(t, T_1, T_2) = 0$ which means $f_{12} = \infty$.

Question 3. (Floating rate annuity)

a) A derivative contract pays $\alpha L_T[T, T+\alpha]$ at time $T+\alpha$. By constructing a portfolio of ZCBs and a libor deposit that replicates the payout, prove that the value at $t \leq T$ of the derivative contract is $Z(t,T) - Z(t,T+\alpha)$.

Solution:

Portfolio	time t	T	$T+\alpha$
1 ZCB with maturity T	$Z(t,T)$	1	$1+\alpha L_T[T,T+\alpha]$
-1 ZCB with maturity $T+\alpha$	$-Z(t,T+\alpha)$		-1
Value	$Z(t,T) - Z(t,T+\alpha)$		$\alpha L_T[T,T+\alpha]$

At time T we take the 1 from the ZCB which matures at T and place it in a libor deposit. At time $T+\alpha$ we receive $1+\alpha L_T[T,T+\alpha]$ from the libor deposit. Since this portfolio has the same payout at maturity as the derivative contract, by the assumption of replicating portfolios the value at t of the derivative contract is $Z(t,T) - Z(t,T+\alpha)$.

b) Let $T_0, T_1, \ldots T_n$ be a sequence of times, with $T_{i+1} = T_i + \alpha$ for a constant $\alpha > 0$. Use your results from (a) to show that a floating leg of libor payments $\alpha L_{T_i}[T_i, T_i+\alpha]$ at times T_{i+1}, $i = 0, 1, \ldots, n-1$, has value at time $t \leq T_0$ equal to a simple linear combination of ZCB prices.

Solution: Receiving libor payments $\alpha L_{T_i}[T_i, T_i+\alpha]$ at times $T_1, T_1, T_2, \ldots, T_n$ has value

$$Z(t,T_0) - Z(t,T_1) + Z(t,T_1) - Z(t,T_2) + \cdots - Z(t,T_n) = Z(t,T_0) - Z(t,T_n).$$

c) Hence find the value of a spot-starting infinite stream of libor payments, that is, when $t = T_0 = 0$ and as $n \to \infty$.

Solution: If $t = T_0 = 0$ then the value is

$$Z(0,0) - Z(0,T_n) = 1 - Z(0,T_n).$$

Since $\lim_{n\to\infty} Z(0,T_n) = 0$ we find that a sport-starting infinite stream of libor payments has value 1.

Question 4. The *interest rate delta* of a derivative contract is defined as the partial derivative, $\frac{\partial}{\partial r}$, of its value, and measures the sensitivity of the price to interest rate changes. Assume that $Z(0,j) = \frac{1}{(1+r)^j}$. Compute the interest rate delta of :

a) An annual Libor stream (as in 3(b), with $\alpha = 1$) starting at $T_0 = 0$ and ending at $T_n = n$.

b) A fixed rate annuity paying c each year from year 1 to year n.

Solution:

a) From 3(b) we have that the value of this contract is $V(t, T_n) = Z(t, T_0) - Z(t, T_n)$. Plugging the given values, this gives us $V(0, T_n) = Z(0, 0) - Z(0, n) = 1 - \frac{1}{(1+r)^n}$. Thus,

$$\frac{\delta V}{\delta r} = \frac{\delta}{\delta r}(1 - \frac{1}{(1+r)^n}) = -n * \frac{-1}{(1+r)^{n+1}} = \frac{n}{(1+r)^{n+1}}.$$

Note that this value is positive, meaning the value of this contract *increases* as interest rates increase, holding all else equal.

b) Recall that the value of this contract is the sum of the discounted payments, c:

$$V(0,n) = \sum_{i=1}^{n} cZ(0,i) = \text{c} \sum_{i=1}^{n} \frac{1}{(1+r)^i}.$$

Taking the partial derivative with respect to r, we find the following:

$$\frac{\delta V}{\delta r} = c \sum_{i=1}^{n} \frac{\delta}{\delta r}(\frac{1}{(1+r)^i}) = c \sum_{i=1}^{n} \frac{-i}{(1+r)^{i+1}} = -c \sum_{i=1}^{n} \frac{i}{(1+r)^{i+1}}.$$

Unlike in (a), note that this delta value is negative, i.e. the value of this contract *decreases* with interest rates, all else equal.