Math 210, Spring 2022

Problem Set # 4

Due February 16, 2022 at 11:59pm on Gradescope

Question 1. Let S_t be the current price of a stock that pays no dividends.

a) Let r_{bid} be the interest rate at which one can invest/lend money, and r_{off} be the interest rate at which one can borrow money, $r_{\text{bid}} \leq r_{\text{off}}$. Both rates are continuously compounded. Using arbitrage arguments, find upper and lower bounds for the forward price of the stock for a forward contract with maturity $T > t$.

b) How does your answer change if the stock itself has bid price $S_{t,\text{bid}}$ and offer price $S_{t,\text{off}}$?

Solution:

a) We guess

$$S_t e^{r_{\text{bid}}(T-t)} \leq F(t,T) \leq S_t e^{r_{\text{off}}(T-t)}$$

and show this using arbitrage arguments. First suppose $F(t,T) > S_t e^{r_{\text{off}}(T-t)}$. In this case the forward price is expensive so we should short the forward contract. Consider the following portfolio:

Portfolio	time t	time T
1 short forward contract at forward price	0	$F(t,T) - S_T$
long 1 share stock	S_t	S_T
cash	$-S_t$	$-S_t e^{r_{\text{off}}(T-t)}$
Value	0	$F(t,T) - S_t e^{r_{\text{off}}(T-t)}$

By assumption $F(t,T) - S_t e^{r_{\text{off}}(T-t)} > 0$ and thus this is an arbitrage portfolio. None exist and so we must have $F(t,T) \leq S_t e^{r_{\text{off}}(T-t)}$.

Now assume $F(t,T) < S_t e^{r_{\text{bid}}(T-t)}$. In this case the forward price is cheap so we should go long a forward contract. Consider the following portfolio:

Portfolio	time t	time T
1 long forward contract at forward price	0	$S_T - F(t,T)$
short 1 share stock	$-S_t$	$-S_T$
cash	S_t	$-S_t e^{r_{\text{bid}}(T-t)}$
Value	0	$-F(t,T) + S_t e^{r_{\text{bid}}(T-t)}$

By assumption $-F(t,T) + S_t e^{r_{\text{bid}}(T-t)} < 0$ and thus this is an arbitrage portfolio. None exist and so we must have $F(t,T) \geq S_t e^{r_{\text{bid}}(T-t)}$.

b) The bid price $S_{t,\text{bid}}$ is the price at which the exchange buys. The offer price $S_{t,\text{off}}$ is the price at which the exchange sells. For stocks we have $S_{t,\text{bid}} \leq S_{t,\text{off}}$. In the first portfolio above you are buying from the exchange and pay the price $S_{t,\text{off}}$. Plugging that value in we find

$$F(t,T) \leq S_{t,\text{off}} e^{r_{\text{off}}(T-t)}.$$

In the second portfolio you are selling stock to the exchange and you receive $S_{t,\text{bid}}$. Plugging in that value we find

$$S_{t,\text{bid}} e^{r_{\text{bid}}(T-t)} \leq F(t,T).$$

Putting both inequalities together we conclude

$$S_{t,\text{bid}} e^{r_{\text{bid}}(T-t)} \leq F(t,T) \leq S_{t,\text{off}} e^{r_{\text{off}}(T-t)}.$$

Question 2. FX forwards are among the most liquid derivative contracts in the world and often reveal more about the health of money markets (markets for borrowing or lending cash) than published short-term interest rates themselves.

a) On Oct. 3, 2008 the euro dollar FX rate was trading at €1 = \$1.3772, and the forward price for a maturity April 3, 2009 forward contract was \$1.3891. Assuming six-month euro interest rates were 5.415%, what is the implied six-month dollar rate? Both interest rates are quoted with act/360 daycount and semi-annual compounding. There are 182 days between Oct. 3, 2008 and April 3, 2009.

b) Published six-month dollar rates were actually 4.13125%. What arbitrage opportunity existed? What transactions does a potential arbitrageur need to to undertake to exploit this opportunity?

c) During the financial crisis, several European commercial banks badly needed to borrow dollar cash, but their only source of funds was euro cash from the European Central Bank (ECB). These banks would: borrow euro cash for six months from the ECB; sell euros/buy dollars in the spot FX market; and sell dollars/buy euros six months forward (to neutralize the FX risk on their euro liability). Explain briefly how these actions may have created the arbitrage opportunity in (b), which existed for several months in late 2008.

Solution:

a) In class we found that the forward price for foreign exchange was $F(t,T) = X_t e^{(r_\$ - r_f)(T-t)}$ for interest compounded continuously. If interest is compounded annually m times per year with accrual factor α then we get $F(t,T) = X_t \left(\frac{1+\alpha r_\$}{1+\alpha r_f}\right)^{m(T-t)}$. For this problem interest is compounded once and $\alpha = 182/360$ so we must solve

$$F(0,1/2) = X_0 \left(\frac{1+\alpha r_\$}{1+\alpha r_f}\right)$$

for r_f. We find $r_\$ = 7.17\%$.

b) A potential arbitrageur could borrow dollars and deposit euros to make a risk free profit. The following portfolio is an arbitrage portfolio. All values are recorded in dollars.

Portfolio	time t=0	time T= 6 months
1 short forward contract at forward price	0	$F(t,T) - X_T$
$1/(1+\alpha r_f)$ of foreign currency	$X_t/(1+\alpha r_f)$	X_T
$-X_t/(1+\alpha r_f)$ dollar currency	$-X_t/(1+\alpha r_f)$	$-X_t \frac{(1+\alpha r_\$)}{(1+\alpha r_f)}$
Value	0	$F(t,T) - X_t \frac{(1+\alpha r_\$)}{(1+\alpha r_f)}$

Plugging in values we find

$$F(t,T) - X_t \frac{(1+\alpha r_\$)}{(1+\alpha r_f)} = 0.0206$$

which is positive therefore this is an arbitrage portfolio.

c) The European commercial banks were undertaking the opposite of the trades we found in part (b). This imbalance in trades could have caused the arbitrage opportunity.

Question 3. Suppose a stock is currently (time $t = 0$) worth 100. Further, suppose the one year annually compounded interest rate is 2%, and the two year annually compounded rate is 3%. Find the following:

a) The forward price for a forward contract on the stock with maturity year $T_1 = 1$.

b) The forward price for a forward contract on the stock with maturity year $T_2 = 2$.

c) The forward price for a forward contract with maturity $T_1 = 1$ on a ZCB with maturity $T_2 = 2$.

d) The forward price for a forward contract with maturity $T_1 = 1$ on a forward contract on the stock with maturity $T_2 = 2$ and delivery price $K = 101$.

Solution:

a) $F(t, T_1) = \frac{St}{Z(t,T_1)} = \frac{100}{1/(1+2\%)} = 102$

b) $F(t, T_2) = \frac{St}{Z(t,T_2)} = \frac{100}{1/(1+3\%)^2} = 106.09$

c) One way to approach this problem is by using non-arbitrage, consider the following portfolio:

Portfolio	$t = 0$	$T_1 = 1$	$T_2 = 2$
1 long such forward contract	0	$-F(t, T_1)$	1
short $Z(t, T_2)$ ZCB's with maturity $T_2 = 2$	-$Z(t, T_2)$		-1
long $Z(t, T_2)$ ZCB's with maturity $T_1 = 1$	$Z(t, T_2)$	$\frac{Z(t,T_2)}{Z(t,T_1)}$	
Value	0	$\frac{Z(t,T_2)}{Z(t,T_1)} - F(t, T_1)$	0

According to Non-arbitrage principle,

$$\frac{Z(t, T_2)}{Z(t, T_1)} - F(t, T_1) = 0$$

which gives us $F(t, T_1) = \frac{100+2\%}{(100+3\%)^2} \approx 0.9614$.

d) Though this problem can also be solved via non-arbitrage, it might be more intuitive to use replication. Consider two portfolios. Portfolio A is just the forward on forward contract. What happens is you will pay $F(t, T_1)$ at $T_1 = 1$ so that you can buy the stock at $T_2 = 2$ with 101. To ensure monotonicity, you will long $F(t, T_1)$ at T_1 with maturity T_2.

Portfolio A	$t = 0$	$T_1 = 1$	$T_2 = 2$
1 long a forward on forward contract	0	$-F(t, T_1)$	$S_{T_2} - 101$
long $F(t, T_1)$ at T_1 with maturity T_2		$F(t, T_1)$	$\frac{F(t,T_1)}{Z(T1,T2)}$
Value	0	0	$S_{T_2} - 101 + \frac{F(t,T_1)}{Z(T1,T2)}$

Portfolio B will consist of a simple forward contract and ZCB's such that it will have the same value as portfolio A in periods T_1 and T_2.

Portfolio B	$t=0$	$T_1=1$	$T_2=2$
long 1 forward on a stock with maturity T_2	0		$S_{T_2}-106.09$
long $5.09 \cdot Z(t,T_2)$ ZCB's with maturity T_2	$5.09 \cdot Z(t,T_2)$		5.09
short $F(t,T_1) \cdot Z(t,T_1)$ ZCB's with maturity T_1	$-F(t,T_1) \cdot Z(t,T_1)$	$-F(t,T_1)$	
long $F(t,T_1)$ at T_1 with maturity T_2		$F(t,T_1)$	$\frac{F(t,T_1)}{Z(T1,T2)}$
Value	$5.09 \cdot Z(t,T_2)$ $-F(t,T_1) \cdot Z(t,T_1)$	0	$S_{T_2}-101+\frac{F(t,T_1)}{Z(T1,T2)}$

By replication, we know that two portfolios will have the same value at t, thus

$$5.09 \cdot Z(t,T_2) - F(t,T_1) \cdot Z(t,T_1) = 0$$

which gives us $F(t,T_1) = \frac{5.09 \cdot Z(t,T_2)}{Z(t,T_1)} \approx 4.8938$.