Math 210, Spring 2022

Problem Set # 1

Due January 26, 2022 at 11:59pm on Gradescope

Question 1. Equivalent interest rates and doubling time.

a) Interest is compounded twice per year for 10 years at a rate of $r_s = 3.00\%$ per annum. Find the equivalent interest rate r_A for annual compounding and r for continuous compounding

b) Find the number of years required for your balance to double if interest is compounded annually at rate r_A, twice per year at rate r_s and continuously at rate r. Give your answer correct to within one-tenth of a year.

c) The *Rule of 72* is a quick tool for estimating doubling times. It says that the doubling time for a rate r is roughly $72/r$. Use the Rule of 72 to estimate the doubling time. How does this compare with your answer from part b?

d) Consider an asset that pays N at maturity 15 years from now. If the present value of the asset is \$300 find N. Compute N for interest compounded annually at a rate r_A and interest compounded continuously at rate r (use the interest rates from part (a)).

Solution. a) Annual compounding:

$$(1 + \tfrac{r_s}{2})^{2*15} = (1 + r_A)^{15}, \text{ and solve for } r_A.$$
$$r_A = .0302$$

Continuous compounding:

$$(1 + \tfrac{r_s}{2})^{2*15} = e^{15r}, \text{ and solve for } r.$$
$$r = .0298$$

b) Annual compounding:

$$2 = (1 + r_A)^T, \text{ and solve for } T.$$
$$T = \frac{ln(2)}{ln(1+r_A)} = 23.4$$

Semi-Annual compounding:

$$2 = (1 + r_2/2)^{2T}, \text{ and solve for } T.$$
$$T = \frac{ln(2)}{2ln(1+r_s/2)} = 23.3$$

Continuous compounding:

$$2 = e^{rT}, \text{ and solve for } T.$$
$$T = 23.1$$

c) Annual compounding:

$$T = \frac{72}{3.02} = 23.8.$$

Semi-Annual compounding:

$$T = \frac{72}{3.00} = 24.$$

Continuous compounding:

$$T = \frac{72}{2.98} = 24.2.$$

Note that all approximations are not far from their true value.

d) Annual compounding:

$$N = 300(1 + r_A)^{15} = 468.92.$$

Continuous compounding:

$$N = 300e^{15r} = 468.92.$$

Question 2. You purchase a home for \$500,000 with a 30-year mortgage and make a 20% down payment. Suppose that you make equal mortgage payments each year and that the annually compounded interest rate is 2.5%.

a) How much are your annual payments?

b) When will the balance remaining on your mortgage be less that $\$200,000$?

Solution. a) Recall the formula for annual payments, c:

$$c = \frac{rV}{1-(1+r)^{-m}},$$

and plug in $r = .025$, $V = 500,000*0.8 = 400,000$, and $m = 30$. We obtain the following:

$$c = \$19,111.06.$$

b) Recall the formula for the remaining balance, B_k, after k years:

$$B_k = \frac{V}{(1+r)^m - 1}((1+r)^m - (1+r)^k),$$

and plug-in same values as part (a) and $B_k = 200,000$. We obtain $k = 17.7$ years. Since payments are annual, after 18 years the balance will be below \$200,000.

Question 3. The current price of a stock paying no income is 50. Assume the annually compounded zero rate will be 3% for the next 2 years.

a) Find the current value of a forward contract on the stock if the delivery price is 55 and maturity is in 2 years.

b) Suppose one year from now the stock is still trading at 50. How much has the value of your forward contract gone up or down from year 0 to year 1? This is sometimes called the *carry* of the trade.

c) Suppose the stock price is 65 in 2 years. Find the value of the forward contract from part (a) to the long counterparty at the maturity time of the contract.

Solution. a) The value of the contract can be expressed as follows:

$$V_{55} = (F(0,2) - 55)Z(0,2),$$

where $Z(0,2) = \frac{1}{1.03^2} = 0.9426$ and $F(0,2) = \frac{S_0}{Z(0,2)} = \frac{50}{0.9426} = 53.045$. Plugging in, we see $V_{55} = -1.84$

b) We can use the same formula as in part (a) while adjusting for the new time period:

$$V_{55} = (F(1,2) - 55)Z(1,2),$$

with $Z(1,2) = \frac{1}{1.03} = 0.971$ and $F(1,2) = \frac{S_1}{Z(1,2)} = \frac{50}{0.971} = 51.5$. Plugging in, we obtain $V_{55} = -3.398$. This is a difference of -1.55.

c) At maturity, the value is simply the difference between the stock price at maturity and the delivery price: $V_{55} = 65 - 55 = 10$.

Question 4. a) At time t you own one stock that pays no dividends, and observe that $F(t,T) < S_t/Z(t,T)$. What arbitrage is available to you, assuming that you can only trade the stock, ZCB and forward contract? Be precise about the transactions you should execute to exploit the arbitrage.

b) Suppose that $S_t = 45$, $F(t,T) = 48$, and $Z(t,T) = 0.89$. How much profit will you earn at time T with your arbitrage portfolio from part (a)?

Solution. a) At time t we long a forward contract at the forward price, short a stock and buy ZCB's worth $S_t/Z(t,T)$. The portfolio is shown below:

Portfolio	time $= t$	time $= T$
1 long forward contract at forward price	0	$S_T - F(t,T)$
short 1 stock	$-S_t$	$-S_T$
$\frac{S_t}{Z(t,T)}$ ZCB's	$\frac{S_t}{Z(t,T)} * Z(t,T) = S_t$	$\frac{S_t}{Z(t,T)}$
Value	0	$\frac{S_t}{Z(t,T} - F(t,T)$

Since $F(t,T) < \frac{S_t}{Z(t,T)}$ the value at time T is positive, thus we have an arbitrage portfolio.

b) Plugging in the given values, we obtain the following:

$$\frac{S_t}{Z(t,T)} - F(t,T) = \frac{45}{0.89} - 48 = 2.5618$$