Math 210, Spring 2022

Problem Set # 2

Due Feb 2, 2022 at 11:59pm on Gradescope

Question 1. The *conditional probability* that event A occurs given that B occurs is denoted $P(A|B)$. A standard, very useful theorem is Bayes' Theorem, which says:

$$P(A|B) = \frac{P(B|A)P(A)}{P(B)}.$$

A new unusual disease is spreading. Currently one in every one million people has it. A company develops a test for the disease which they say is 99% accurate, in the sense that a person with the disease will test positive 99% of the time, and a person who doesn't have it will test negative 99% of the time.

a) Write what it means for the test to be 99% accurate as two conditional probabilities. Use the events *test positive, test negative, sick, not sick.*

b) If you test positive, what are the odds you have the disease? *Hint: first write this as a conditional probability, then use Bayes' Theorem.*

Solution:

a) $P(testpositive|sick) = .99$ and $P(testnegative|notsick) = .99$.

b) We're solving for $P(sick|testpositive)$. By Bayes' Theorem,

$$P(sick|testpositive) = \frac{P(testpositive|sick)P(sick)}{P(testpositive)}.$$

So far, we know that $P(testpositive|sick) = .99$, and $P(sick) = 10^{-6}$ (one in one million). We can write

$$\begin{aligned} P(testpositive) &= P(testpositive \cap sick) + P(testpositive \cap notsick) \\ &= 10^{-6} \cdot 0.99 + (1 - 0.99) \cdot (1 - 10^{-6}). \end{aligned}$$

Putting this together, we get that $P(sick|testpositive) \equiv 9.9 \cdot 10^{-5}$, or about one in ten thousand.

Question 2. Suppose, instead of a standard annuity, you have the following cashflow:

- A payment of 100 at the end of every first quarter, and

- A payment of 300 at the end of every year.

This lasts for 5 years, i.e. 10 total payments. What is the present value of this cashflow at the start of the first year if the quarterly compounded annual interest rate is 3%? You may assume a 30/360 daycount.

Solution:

Present Value of 100s:

$$\sum_{n=0}^{4} \frac{100}{(1+\frac{3\%}{4})^{4n+1}} = 467.899$$

Present Value of 300s:

$$\sum_{n=1}^{5} \frac{300}{(1+\frac{3\%}{4})^{4n}} = 1372.583$$

In total, the present value is

$$467.90 + 1372.58 = 1840.482$$

Question 3. Two market standards for US dollar interest rates are semi-annual compounding with 30/360 daycount (semi-bond, denoted y_{sb}) and annual compounding with act/360 daycount (annual-money, denoted y_{am}). *Note: it's important to know how to do this math, but you won't need to know these industry terms on exams*

a) Derive an expression for y_{am} in terms of y_{sb}. You may assume all years have 365 days.

b) Calculate y_{am} when $y_{sb} = 5\%, 6\%$, and 7%.

Solution:

a) Both interest rates must give the same accrued interest for one year. This implies

$$\left(1 + y_{am}\frac{365}{360}\right) = \left(1 + y_{sb}\frac{180}{360}\right)^2 = \left(1 + y_{sb}\frac{1}{2}\right)^2.$$

Solving for Y_{am} we find.

$$y_{am} = \frac{18}{73} y_{sb}(y_{sb} + 4).$$

b)

$$y_{am}(5\%) = 4.993\%$$
$$y_{am}(6\%) = 6.007\%$$
$$y_{am}(7\%) = 7.025\%.$$

Question 4. Suppose the continuously compounded annual interest rate is 4%. Find

a) The equivalent annually compounded rate with act/360 daycount.

b) The equivalent semiannually compounded rate with act/365 daycount. Assume quarters have lengths 92, 92, 91, 90 days.

Solution:

a) The equivalent annually compounded rate with act/360 daycount r_1 satisfies

$$e^{4\%} = 1 + \frac{365}{360} r_1$$

solve for r_1, $r_1 = 4.025\%$

b) The equivalent semiannually compounded rate with act/365 daycount r_2 satisfies

$$e^{4\%} = (1 + \frac{184}{365} r_2)(1 + \frac{181}{365} r_2)$$

solve for r_2, $r_2 = 4.040\%$

Question 5. A *simple interest rate* of r for T years means a 100 investment becomes $100(1 + rT)$ at maturity T.

a) For simple interest 5% for ten years, calculate the equivalent compound interest rate with (i) annual, (ii) quarterly and (iii) continuous compounding. Assume 30/360 daycount.

b) Show that if simple interest of r for T years is equivalent to r^* interest rate with annual compounding, then $r^* \to 0$ as $T \to \infty$.

Solution:

a) Let $q = 5/100 = .05$.

(i) To find the annual rate r we solve

$$1 + qT = \left(1 + \frac{360}{360} r\right)^T$$

for r and find

$$r = (1 + qT)^{1/T} - 1.$$

Plugging in values we get $r = .0414$.

(ii) To find the quarterly rate r we solve

$$1 + qT = \left(1 + \frac{90}{360}r\right)^{4T}$$

for r and find

$$r = 4\left((1 + qT)^{1/(4T)} - 1\right).$$

Plugging in values we get $r = .0408$.

(ii) To find the continuous rate r we solve

$$1 + qT = e^{rT}$$

for r and find

$$r = \frac{1}{T}\ln(1 + qT).$$

Plugging in values we get $r = .0405$.

b) We have the equation

$$1 + rT = (1 + r^*)^T.$$

We can solve this equation for r^* to find

$$r^* = -1 + (1 + rT)^{1/T}.$$

Now taking limits we get

$$\begin{aligned}\lim_{T\to\infty} r^* &= \lim_{T\to\infty} -1 + (1 + rT)^{1/T} \\ &= -1 + \lim_{T\to\infty}(1 + rT)^{1/T} \\ &= -1 + 1 = 0.\end{aligned}$$

To evaluate the limit we use L'Hospital's Rule as follows

$$\begin{aligned}\lim_{T\to\infty}(1 + rT)^{1/T} &= \lim_{T\to\infty} e^{\frac{1}{T}\ln(1+rT)} \\ &= \exp\left(\lim_{T\to\infty}\frac{1}{T}\ln(1 + rT)\right) \\ &= \exp\left(\lim_{T\to\infty}\frac{1}{1}\frac{1}{1 + rT}r\right) \\ &= e^0 = 1.\end{aligned}$$